AN ENHANCED EULER CHARACTERISTIC OF SUTURED INSTANTON HOMOLOGY

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Abstract. For a balanced sutured manifold \((M, \gamma)\), we construct a decomposition of \(\text{SHI}(M, \gamma)\) with respect to torsions in \(H = H_1(M; \mathbb{Z})\), which generalizes the decomposition of \(I^Y\) in previous work of the authors. This decomposition can be regarded as a candidate for the counterpart of the torsion spin\(^c\) decompositions in \(\text{SFH}(M, \gamma)\). Based on this decomposition, we define an enhanced Euler characteristic \(\chi_{en}(\text{SHI}(M, \gamma)) \in \mathbb{Z}[H]/\pm H\) and prove that \(\chi_{en}(\text{SHI}(M, \gamma)) = \chi(\text{SFH}(M, \gamma))\). This provides a better lower bound on \(\dim_{\mathbb{C}} \text{SHI}(M, \gamma)\) than the graded Euler characteristic \(\chi_{gr}(\text{SHI}(M, \gamma))\). As applications, we prove instanton knot homology detects the unknot in any instanton L-space and show that the conjecture \(KHI(Y, K) \cong HFK(Y, K)\) holds for all \((1,1)\)-L-space knots and constrained knots in lens spaces, which include all torus knots and many hyperbolic knots in lens spaces.

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1. Introduction

Sutured instanton Floer homology was introduced by Kronheimer and Mrowka in [KM10b]. It combines instanton Floer homology with sutured manifold theory and has become a powerful tool since then. In [Li19], Ghosh and the first author constructed a decomposition of the sutured instanton Floer homology $\text{SHI}(M, \gamma)$ of a balanced sutured manifold with respect to the group $(H_2(M, \partial M; \mathbb{Z}))^* \cong H_1(M; \mathbb{Z})/\text{Tors}$. More precisely, a basis of $H_2(M, \partial M; \mathbb{Z})$ induces a multi-grading on $\text{SHI}(M, \gamma)$ which can be identified with the group $H_1(M; \mathbb{Z})/\text{Tors}$. In [LY21], the authors of the current paper studied the Euler characteristics of this decomposition of $\text{SHI}(M, \gamma)$ and related it to the Euler characteristic of $\text{SFH}(M, \gamma)$, which is known as the sutured Floer homology introduced by Juhász, and whose Euler characteristic has been understood by work of Friedl, Juhász, and Rasmussen in [FJR09]. The study of Euler characteristics was further used by the author to compute the instanton Floer homology of some families of $p_1$ knots in a general lens space and was recently further utilized by Zhang and Xie [XZ21] to prove that links in $S^3$ all admit irreducible $SU(2)$ representations except for connected sums of Hopf links.

However, only having the decomposition of $\text{SHI}(M, \gamma)$ along the group $H_1(M; \mathbb{Z})/\text{Tors}$ is not fully satisfactory for the following two reasons.

1. Among all known Floer homology theories for sutured manifolds, we have known that sutured monopole Floer homology is isomorphic to sutured Floer homology by work of Lekili [Lek13] and Baldwin and Sivek [BS20], and it is conjectured that the sutured instanton Floer homology is also isomorphic to sutured Floer homology by Kronheimer and Mrowka [KM10b]. However, sutured Floer homology decomposes along spin$^c$ structures and, in particular, the first Chern classes of torsion spin$^c$ structures have Poincaré dual in the torsion part of the group $H_1(M; \mathbb{Z})$, so there should be some corresponding decomposition of sutured instanton Floer homology.

2. The original decomposition along $H_1(M; \mathbb{Z})/\text{Tors}$ collapses all torsion parts into a single summand of $\text{SHI}(M, \gamma)$, and some information may lost in this collision; see Example 1.4.

In this paper, in order to solve this problem coming from collapsing torsion parts, we obtain the following.

**Theorem 1.1 (Main theorem).** Suppose $(M, \gamma)$ is a balanced sutured manifold and $H = H_1(M; \mathbb{Z})$. Then there is a (noncanonical) decomposition

$$\text{SHI}(M, \gamma) = \bigoplus_{h \in H} \text{SHI}(M, \gamma, h).$$

This decomposition depends on some auxiliary choices. In particular, it is defined up to a global shift of $H$. We define the **enhanced Euler characteristic** of $\text{SHI}$ by

$$\chi_{\text{en}}(\text{SHI}(M, \gamma)) := \sum_{h \in H} \chi(\text{SHI}(M, \gamma, h)) \cdot h \in \mathbb{Z}[H]/\pm H.$$

Then we have

$$\chi_{\text{en}}(\text{SHI}(M, \gamma)) = \chi(\text{SFH}(M, \gamma)) \in \mathbb{Z}[H]/\pm H.$$

The similar results also hold for $\text{SHM}(M, \gamma)$.

**Remark 1.2.** If $H_1(M; \mathbb{Z})$ has no torsion, then the decomposition in Theorem 1.1 is just induced by the multi-grading mentioned above and the equation (1.1) reduces to [LY21, Theorem 1.2]. By results in [FJR09], we have $\chi(\text{SFH}(M, \gamma)) = \tau(M, \gamma)$, where $\tau(M, \gamma)$ is a (Turaev-type) torsion element that can be calculated by Fox calculus. In particular, if $\partial M$ consists of tori and $\gamma$ consists
of two parallel copies of a curve \( m_i \) with opposite orientations on each boundary component, by [FJR09 Proof of Lemma 6.1] and [RR17 Proposition 2.1], we have

\[
\tau(M, \gamma) = \tau(M) \cdot \prod_i ([m_i] - 1),
\]

where \( \tau(M) \) is the Turaev torsion of \( M \) [Tur02].

Though the decomposition in Theorem 1.1 is not canonical, we expect it to be well-defined up to a global grading shift of \( H \). The following theorem indicates this decomposition is a candidate for the counterpart of the spin\(^c\) decomposition. The proof is essentially due to [Lek13, BS20].

**Theorem 1.3.** Suppose \((M, \gamma)\) is a balanced sutured manifold. Then there is a spin\(^c\) structure \( s_0 \in \text{Spin}^c(M, \gamma) \) such that for any \( s \in \text{Spin}^c(M, \gamma) \), we have an isomorphism

\[
SHM(M, \gamma, PD(s - s_0)) \cong SFH(M, \gamma, s) \otimes \Lambda,
\]

where \( PD : H^2(M, \partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \) is the Poincaré duality map.

For an element in a group ring \( \mathbb{Z}[G] \)

\[
x = \sum_{g \in G} c_g \cdot g, \text{ for } c_g \in \mathbb{Z},
\]

define

\[
\|x\| = \sum_{g \in G} |c_g|.
\]

This definition is still well-defined for an element in \( \mathbb{Z}[G]/\pm G \). By construction of Euler characteristics, we have

\[
\dim \mathbb{C} SHI(M, \gamma) \geq \|\chi_{en}(SHI(M, \gamma))\| \geq \|\chi_{gr}(SHI(M, \gamma))\|.
\]

To provide an example that the second inequality in (1.2) is not always sharp, and hence \( \chi_{en} \) contains more information than \( \chi_{gr} \), we consider an example from constrained knots studied by the second author [Ye21].

**Example 1.4.** Consider the 1-cusped hyperbolic manifold \( M = m006 \) in the Snappy program [CDMW21]. We have \( H_1(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_{\delta} \cong \mathbb{Z}(t, \tau)/(5\tau) \). By the list of constrained knots in [Ye20], Dehn filling along the slope \((1, 0)\) (in the basis from Snappy) gives the lens space \( L(5, 3) \) and the core knot is the constrained knot \( C(5, 3, 4, 3, 1) \). Suppose \( \gamma \) consists of two parallel copies of the curve of slope \((1, 0)\). Then we have

\[
\tau(M, \gamma) = 1 + r + t + rt + r^2t - r^3t - r^4t + rt^2 + r^2t^2,
\]

and

\[
\tau(M, \gamma)|_{r=1} = 1 + 1 + t + t - t - t^2 + t^2 = 2 + 2t^2.
\]

Hence we have

\[
\|\chi_{en}(SHI(M, \gamma))\| = \|\tau(M, \gamma)\| = 9 \text{ and } \|\chi(SHI(M, \gamma))\| = \|\tau(M, \gamma)\|_{r=1} = 5.
\]

Suppose \( K \) is a knot in a closed 3-manifold \( Y \). Let

\[
Y(1) := Y \setminus B^3 \text{ and } Y(K) := Y \setminus \text{int}N(K).
\]

Suppose \( \delta \) is a simple closed curve on \( \partial Y(K) \cong S^2 \) and suppose \( \gamma_K \) is two copies of the meridian of \( K \) with opposite orientations. In Heegaard Floer theory, we have

\[
SFH(Y(1), \delta) \cong \widehat{HF}(Y) \text{ and } SFH(M, \gamma_K) \cong \widehat{HFK}(Y, K),
\]
where $\hat{HF}(Y)$ and $\hat{FK}$ are the hat versions of the Heegaard Floer homology and the knot Floer homology defined by Oszváth and Szabó [OS04c, OS04a, Ras03]. In instanton theory, Kronheimer and Mrowka [KM10b] defined

$$I^2(Y) := SHI(Y(1), \delta)$$
and

$$KHI(Y, K) := SHI(Y(K), \gamma_K).$$

An application of Theorem 1.1 is the following unknot detection result in rational homology spheres.

**Theorem 1.5.** Suppose $K$ is a null-homologous knot in a rational homology sphere $Y$. If

$$\dim_C I^2(Y) = |H_1(Y; \mathbb{Z})|,$$ (1.3)

then $K$ is the unknot i.e., it bounds a disk in $Y$, if and only if

$$\dim_C KHI(Y, K) = \dim_C I^2(Y).$$ (1.4)

**Remark 1.6.** Since instanton theory is closely related to $SU(2)$ representations of fundamental groups, Theorem 1.5 may be used to show that for any nontrivial null-homologous knot $K$ in a rational homology sphere $Y$, the fundamental group $\pi_1(Y(K))$ admits an irreducible representation in $SU(2)$ such that the meridian of $K$ is mapped to a traceless element in $SU(2)$. However, the authors do not know how to prove the nondegeneracy results similar to [BS19, Section 4.1] for generators of $KHI(Y, K)$.

**Remark 1.7.** Rational homology spheres that satisfy (1.3) are called **instanton L-spaces**. Theorem 1.5 cannot be generalized to knots that are not null-homologous because simple knots in lens spaces also satisfy (1.4) [LY20, Proposition 1.7]. It is a natural conjecture that simple knots are the only knots in lens spaces satisfy (1.4) (For Heegaard Floer theory, see [BGH08, Conjecture 1.5]).

Following the similar strategy, we can prove the following theorem for knots whose $KHI$ have small dimensions.

**Theorem 1.8.** Suppose $K$ is a null-homologous knot in a rational homology sphere $Y$. If

$$\dim_C KHI(Y, K) = \dim_C I^2(Y) + 2 = |H_1(Y; \mathbb{Z})| + 2,$$ (1.5)

then $K$ must be a genus-one-fibred knot.

**Remark 1.9.** The only knots in $S^3$ satisfying (1.5) are the trefoil and its mirror. Hence Theorem 1.8 is a generalization of the trefoil detection result in $S^3$ [BS18, Theorem 1.6]. Both proofs are based on the nonvanishing result on the ‘next-to-top’ grading [BS18, Theorem 1.7] for fibred knots.

**Remark 1.10.** For a knot $K$ in an instanton L-space $Y$ with

$$\dim_C KHI(Y, K) = \dim_C I^2(Y) + 4 = |H_1(Y; \mathbb{Z})| + 4,$$

we may still conclude $K$ is fibred using the same strategy. However, it is impossible to pin down the genus because there are at least two knots in $S^3$ with different genera: the figure-8 knot with genus one and the $T_{(2,5)}$ torus knot with genus two. Recently, there are many new results [BHS21, NZ21] about the Khovanov homology and the knot Floer homology of $T_{(2,5)}$. It is an interesting question that if these results can be applied to instanton knot homology.

**Remark 1.11.** We can prove similar detection results in Heegaard Floer theory; see Section 6.

Another application of Theorem 1.1 is to compute $KHI(Y, K)$ for all $(1,1)$-L-space knots and constrained knots in lens spaces. The calculation is based on the following theorem.
Theorem 1.12 ([LY20, Theorem 1.4], see also [BLY20, Theorem 1.1]). Suppose \( K \subset Y \) is a (1, 1)-knot in a lens space (including \( S^3 \)). Then we have
\[
\dim_c KHI(Y, K) \leq \dim_{\mathbb{F}_2} \widehat{HFK}(Y, K).
\]

Corollary 1.13. Suppose \( K \subset Y \) is a (1, 1)-knot in a lens space (including \( S^3 \)). If \( K \) is either a L-space knot, or a constrained knot, then
\[
\dim_c KHI(Y, K) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y, K).
\]

Proof. The theorem follows from comparing the upper bound from Theorem 1.12 and the lower bound from \( \|\chi_{\text{en}}(KHI(Y, K))\| \). By [RR17, Lemma 3.2] and [GLV18, Theorem 2.2] for L-space knots, and by [Ye21, Section 4] for constrained knots, the upper bound matches the lower bound. \( \square \)

Remark 1.14. In the authors’ previous work [LY21, Corollary 1.9], we proved Corollary 1.13 in the case where \( H_1(Y(K); \mathbb{Z}) \cong \mathbb{Z} \). This is because the lower bound from \( \|\chi_{\text{en}}(KHI(Y, K))\| \) may not equal to the upper bound in [LY20] when \( H_1(Y(\text{int}N(K); \mathbb{Z})) \) has torsions (c.f. Example 1.4).

Remark 1.15. The result in Corollary 1.13 can be generalized to other (1, 1)-knot \( K \) whose \( \widehat{HFK} \) is totally determined by \( \chi(\widehat{HFK}(Y, K)) \), such as \( (\pm 2, p, q) \) pretzel knots for odd integers \( p \) and \( q \) (c.f. [GMM05, Section 5]).

Any lens space \( Y \) has a standard Heegaard splitting of genus 1. A knot \( K \) in a lens space \( Y \) is called a torus knot if \( K \) can be isotoped to lie on the Heegaard torus of \( Y \). This definition generalizes the usual torus knots in \( S^3 \). A knot \( K \) is called a satellite knot if \( Y(\text{int}N(K)) \) has an essential torus. A knot \( K \) is called hyperbolic if \( Y(K) \) admits a hyperbolic metric of finite volume. By Thurston’s Hyperbolization Theorem for Haken 3-manifolds, we have a good classification of knots in lens spaces.

Proposition 1.16 ([Wil09, Proposition 3.1]). Suppose \( K \) is a knot in a lens space \( Y \). If \( Y(K) \) is irreducible, then \( K \) is either a torus knot, a satellite knot, or a hyperbolic knot.

It is straightforward to check that torus knots are (1, 1)-knots, and their complements are Seifert fibred spaces.

Proposition 1.17 ([RR17, Theorem 5.1]). Knots in lens spaces with Seifert fibred complements are L-space knots.

Combining Proposition 1.17 and Corollary 1.13, we have the following result.

Corollary 1.18. For any torus knot \( K \) in a lens space \( Y \), we have
\[
\dim_c KHI(Y, K) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y, K).
\]

Complements of many constrained knots are orientable hyperbolic 1-cusped manifolds with simple ideal triangulations. In particular, among 286 orientable 1-cusped manifolds that have ideal triangulations with at most five ideal tetrahedra, there are 232 manifolds that are complements of constrained knots. More examples can be found in [Ye20]. Indeed, [Ye21, Conjecture 2] conjectured that most constrained knots are hyperbolic knots.
1.1. Organization and sketch of the proofs.

Suppose \((M, \gamma)\) is a balanced sutured manifold. To deal with \(\text{SH}(M, \gamma)\) and \(\text{SH}_{\text{I}}(M, \gamma)\) together, we use formal sutured homology \(\text{SH}(M, \gamma)\) constructed in [LY21]. Its construction is based on the following three axioms:

- (A1) the adjunction inequality axiom;
- (A2) the surgery exact triangle axiom;
- (A3) the canonical \(\mathbb{Z}_2\)-grading axiom.

These axioms are motivated by the necessary properties of a Floer theory that are used to build a sutured homology for balanced sutured manifolds. A \((3+1)\)-TQFT is called a Floer-type theory if it satisfies (A1)-(A3). For example, instanton theory, monopole theory, and Heegaard Floer theory are all Floer-type theories and the corresponding formal sutured homologies of \((M, \gamma)\) are denoted by

\[
\text{SH}(M, \gamma), \, \text{SH}(M, \gamma)_{\text{I}}, \, \text{and} \, \text{SHF}(M, \gamma),
\]

respectively. The former two are not new and we have

\[
\text{SH}_{\text{I}}(M, \gamma) = \text{SH}(M, \gamma) \text{ and } \text{SH}(M, \gamma) \otimes \Lambda \cong \text{SH}(M, \gamma).
\]

The last one \(\text{SHF}(M, \gamma)\) was used to build a bridge between \(\text{SH}(M, \gamma)\) and \(\text{SFH}(M, \gamma)\) in [LY21] and we have

\[
\text{SHF}(M, \gamma) \cong \text{SFH}(M, \gamma).
\]

In Section 2 we review the construction of \(\text{SH}(M, \gamma)\), the gradings on \(\text{SH}(M, \gamma)\) associated to admissible surfaces, and the maps on \(\text{SH}(-M, -\gamma)\) associated to contact handle attachments (called contact gluing maps). Since we will use contact gluing maps frequently, it is more convenient to consider \((-M, -\gamma)\), the sutured manifold with the reverse orientation. Hence we state all results with a minus sign.

In Section 3 we generalize the decomposition associated to a rationally null-homologous knot in [LY20, Section 4] to a connected rationally null-homologous tangle \(\alpha\), i.e. \([\alpha] = 0 \in H_1(M, \partial M; \mathbb{Q})\). We write \(M_\alpha = M \setminus \text{int} N(\alpha)\). Then by Lemma 3.19 we have

\[
\text{rk}_Z H_1(M_\alpha; \mathbb{Z}) = \text{rk}_Z H_1(M; \mathbb{Z}) + 1,
\]

and there is a surjective map

\[
H_1(M_\alpha; \mathbb{Z}) \to H_1(M; \mathbb{Z})/[m_\alpha] \cong H_1(M; \mathbb{Z}),
\]

where \(m_\alpha\) is the meridian of \(\alpha\). Moreover, after picking some suitable \(\alpha\), the pre-images of some torsions in \(H_1(M; \mathbb{Z})\) are distinguished in the free part of \(H_1(M_\alpha; \mathbb{Z})\). Since the difference in the free part can be detected by the gradings associated to admissible surfaces, we can decompose \(\text{SH}(-M, -\gamma)\) by considering direct summands of \(\text{SH}(-M_\alpha, -\gamma \cup m_\alpha)\) in some gradings whose total dimension is the same as that of \(\text{SH}(-M, -\gamma)\). The direct sum of these summands are denoted by \(\mathcal{S}(\alpha)(-M, -\gamma)\), which generalizes \(\mathcal{I}_+(-\hat{Y}, \hat{K})\) in [LY20, Section 4].

The above method can be applies iteratively for a tangle \(T\) with more than one component and finally we can distinguish all torsions in \(H_1(M; \mathbb{Z})\) by choosing \(T\) such that \(H_1(M_T; \mathbb{Z})\) is torsion-free (c.f. Lemma 3.20). Similarly, we can identify \(\text{SH}(-M, -\gamma)\) with a direct summand of \(\text{SH}(-M_T, -\gamma \cup m_T)\), where \(m_T\) is the union of meridians of tangle components of \(T\). Since \(H_1(M_T; \mathbb{Z})\) has no torsion, all torsions that are mixed on \(H_1(M; \mathbb{Z})\) can be distinguished on \(H_1(M_T; \mathbb{Z})\), and this provides the desired decomposition in Theorem 1.1.

Suppose \(j_\ast\) is the map on group rings induced by

\[
j: H_1(M_T; \mathbb{Z}) \to H_1(M; \mathbb{Z}).
\]
Given the construction of $\mathcal{SH}_T(-M, -\gamma)$, the equation \[ (1.6) \] \[
\chi_{en}(\mathcal{SH}(-M, -\gamma)) := j_* (\chi_{gr}(\mathcal{SH}_T(-M, -\gamma))) = \chi(\mathcal{SFH}(-M, -\gamma)). \]
We prove this equation by introducing a decomposition $\mathcal{SFH}_T(-M, -\gamma)$ of $\mathcal{SFH}(-M, -\gamma)$ similar to $\mathcal{SH}_T(-M, -\gamma)$. However, the construction of $\mathcal{SFH}$ is based on balanced diagrams of balanced sutured manifolds, which is different from the construction of $\mathcal{SH}$ by closures. So we have to show that $\mathcal{SFH}$ satisfies the similar setups of $\mathcal{SH}$ to construct $\mathcal{SFH}_T(-M, -\gamma)$. This is the main goal of Section 4, where we collect results for $\mathcal{SFH}$ parallel to $\mathcal{SH}$, including gradings associated to admissible surfaces, the surgery exact triangle, the bypass exact triangle, and contact gluing maps.

Since $H_1(M_T; \mathbb{Z})$ is torsion-free, we can apply [LY21, Theorem 1.2] to obtain
\[ (1.7) \]
\[
\chi_{gr}(\mathcal{SH}_T(-M, -\gamma)) = \chi_{gr}(\mathcal{SFH}_T(-M, -\gamma)).
\]
By discussion on spin$^c$ structures, we show
\[ (1.8) \]
\[
j_*(\chi_{gr}(\mathcal{SFH}_T(-M, -\gamma))) = \chi(\mathcal{SFH}(-M, -\gamma)).
\]
Equations \[ (1.7) \] and \[ (1.8) \] imply the equation \[ (1.6) \], which finishes the proof of Theorem 1.1. The detailed proof can be found in Section 5.

The proof of Theorem 1.3 is almost straightforward, based on the work of Lekili [Lek13] and Baldwin and Sivek [BS20]. Since $\mathcal{SH}_T(-M, -\gamma)$ is direct summands of $\mathcal{SH}(-M_T, -\gamma \cup m_T)$ in some gradings, it suffices to prove the theorem when $H_1(M_T; \mathbb{Z})$ is torsion-free. In this case, the decomposition is just induced by admissible surfaces and Theorem 1.3 follows from the isomorphism
\[
\mathcal{SH}_M(M, \gamma) \cong \mathcal{SFH}(M, \gamma) \otimes \Lambda.
\]
The detailed proof can be also found in Section 5.

In Section 6, we study knots whose $\text{dim}_\mathbb{C} \mathcal{KH}^T$ are small and prove Theorem 1.5, Theorem 1.8, and analog theorems in Heegaard Floer theory.

1.2. Conventions. If it is not mentioned, all manifolds are smooth, oriented, and connected. Homology groups and cohomology groups are with $\mathbb{Z}$ coefficients, i.e. $H_*(M) := H_*(M; \mathbb{Z})$ for any manifold $M$. For other coefficients (like $\mathbb{Q}$), we still write $H_*(M; \mathbb{Q})$. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. For a simple closed curve on a surface, we do not distinguish between its homology class and itself. The algebraic intersection number of two curves $\alpha$ and $\beta$ on a surface is denoted by $\alpha \cdot \beta$, while the number of intersection points between $\alpha$ and $\beta$ is denoted by $|\alpha \cap \beta|$. A basis $(m, l)$ of $H_1(T^2; \mathbb{Z})$ satisfies $m \cdot l = -1$. The surgery means the Dehn surgery and the slope $q/p$ in the basis $(m, l)$ corresponds to the curve $qm + pl$.

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2. Formal sutured homology

In this section, we describe the construction of formal sutured homology $\mathcal{SH}$ introduced in [LY21] and collect useful properties.
2.1. Notations of formal sutured homology.

Definition 2.1 ([Juh06, KM10b]). A balanced sutured manifold \((M, \gamma)\) consists of a compact oriented 3-manifold \(M\) with non-empty boundary together with a closed 1-submanifold \(\gamma\) on \(\partial M\). Let \(A(\gamma) = \{-1, 1\} \times \gamma\) be an annular neighborhood of \(\gamma \subset \partial M\) and let \(R(\gamma) = \partial M \setminus \text{int}(A(\gamma))\). They satisfy the following properties.

(1) Neither \(M\) nor \(R(\gamma)\) has a closed component.
(2) If \(\partial A(\gamma) = \partial R(\gamma)\) is oriented in the same way as \(\gamma\), then we require this orientation of \(\partial R(\gamma)\) induces one on \(R(\gamma)\). The induced orientation on \(R(\gamma)\) is called the canonical orientation.
(3) Let \(\tilde{R}_A(\gamma)\) be the part of \(R(\gamma)\) so that the canonical orientation coincides with the induced orientation on \(\partial M\), and let \(\tilde{R}_A(\gamma) = R(\gamma) \setminus \tilde{R}_A(\gamma)\). We require that \(\chi(R_+(\gamma)) = \chi(R_-(\gamma))\). If \(\gamma\) is clear in the contents, we simply write \(\tilde{R}_A(\gamma) = R(\gamma)\).

Suppose \(H\) is a \((3+1)\)-TQFT, i.e. a monoidal functor
\[ H : \text{Cob}^{3+1} \to \text{Vect}, \]
where \(F\) is a field. Moreover, suppose \(H\) satisfies the axioms (A1)-(A3) in [LY21] Section 2.1, then we called \(H\) a Floer-type theory as in [LY21] Definition 2.1. For such \(H\), we can construct a formal Floer homology for a balanced sutured manifold \((M, \gamma)\), which we denote by \(SH(M, \gamma)\), as in [LY21] Definition 2.17.

The construction was originated from [KM10b], where Kronheimer and Mrowka carried out the construction with monopole and instanton theories. After that, several groups of people studies these Floer homologies for balanced sutured manifolds extensively, see for example [BS16a, BS18, Li19, GL19, Wan20]. In [LY21], we wrote down a set of axioms and showed that all these constructions could be adapted to depend only on the axioms (A1), (A2), and (A3) and hence work for the formal Floer homology \(SH(M, \gamma)\). In what follows, we will briefly summarize the definitions and results that will be needed for the purpose of the current paper, while readers are referred to [LY21] for more detailed discussions.

In particular, instanton theory, monopole theory, and Heegaard Floer theory are special examples of Floer-type theories. Experts may check Table 1 for notations of formal sutured homology based on various Floer theories.

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<td>Sutured homology</td>
<td>(SH)</td>
<td>(SHI)</td>
<td>(SHM)</td>
<td>(SHF)</td>
</tr>
</tbody>
</table>

For closed 3-manifolds and knots with basepoints, we can construct balanced sutured manifolds and then apply the formal sutured homology as follows.

Definition 2.2. Suppose that \(Y\) is a closed 3-manifold and \(z \in Y\) is a basepoint. Let \(Y(1)\) be obtained from \(Y\) by removing a 3-ball containing \(z\) and let \(\delta\) be a simple closed curve on \(\partial Y(1) \cong S^2\). Suppose that \(K \subset Y\) is a knot and \(w\) is a basepoint on \(K\). Let \(Y(K)\) be the knot complement of
and let $\gamma = m \cup (-m)$ consist of two meridians with opposite orientations of $K$ near $w$. Then $(Y(1), \delta)$ and $(Y(K), \gamma)$ are balanced sutured manifolds. Define

$$\tilde{H}(Y, z) := \text{SH}(Y(1), \delta) \text{ and } \text{KH}(Y, K, w) := \text{SH}(Y(K), \gamma).$$

**Convention.** Different choices of the basepoints give isomorphic vector spaces. Since in the current paper we only care about the isomorphism class of the vector spaces, we omit the basepoints and simply write $\tilde{H}(Y)$ and $\text{KH}(Y, K)$ instead.

### 2.2. Gradings and Euler characteristics.

If $S$ is a properly embedded surface in $M$ with some admissible conditions, the first author \[Li19\] constructed a grading on $\text{SH}(M, \gamma)$ for monopole theory and instanton theory (with some pioneering work done by Kronheimer and Mrowka \[KM10a\] and Baldwin and Sivek \[BS18\]). This construction can be adapted to the formal sutured homology associated to any other Floer-type theory.

**Definition 2.3 (GL19).** Suppose $(M, \gamma)$ is a balanced sutured manifold and $S \subset M$ is a properly embedded surface. The surface $S$ is called an **admissible surface** if the followings hold.

1. Every boundary component of $S$ intersects $\gamma$ transversely and nontrivially.
2. $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is an even integer.

**Theorem 2.4 ([LY21, Theorem 2.30], see also [Li19, Li18]).** Suppose $(M, \gamma)$ is a balanced sutured manifold and $S \subset (M, \gamma)$ is an admissible surface. Then there is a $\mathbb{Z}$-grading on $\text{SH}(M, \gamma)$ induced by $S$, which we write as

$$\text{SH}(M, \gamma) = \bigoplus_{i \in \mathbb{Z}} \text{SH}(M, \gamma, S, i).$$

This decomposition satisfies the following properties.

1. Suppose $n = \frac{1}{2}|S \cap \gamma|$. If $|i| > \frac{1}{2}(n - \chi(S))$, then $\text{SH}(M, \gamma, S, i) = 0$.
2. If there is a sutured manifold decomposition $(M, \gamma) \xrightarrow{\S} (M', \gamma')$ in the sense of Gabai \[Gab83\], then we have

$$\text{SH}(M, \gamma, S, \frac{1}{2}(n - \chi(S))) \cong \text{SH}(M', \gamma').$$

3. For any $i \in \mathbb{Z}$, we have

$$\text{SH}(M, \gamma, S, i) = \text{SH}(M, \gamma, -S, -i).$$

4. For any $i \in \mathbb{Z}$, we have

$$\text{SH}(M, -\gamma, S, i) \cong \text{SH}(M, \gamma, S, -i).$$

5. For any $i \in \mathbb{Z}$, we have

$$\text{SH}(-M, \gamma, S, i) \cong \text{Hom}_{\mathbb{Z}}(\text{SH}(M, \gamma, S, -i), \mathbb{F}).$$

Based on the term (2) in Theorem 2.4, we can show formal sutured homology detects the tautness of balanced sutured manifolds.

**Definition 2.5 ([Juh06]).** A sutured manifold $(M, \gamma)$ is called **taut** if $M$ is irreducible and $R_+(\gamma)$ and $R_-(\gamma)$ are both incompressible and Thurston norm-minimizing in the homology class that they represent in $H_2(M, \gamma)$.

**Theorem 2.6 ([Juh06, Juh08, KM10b]).** Suppose $(M, \gamma)$ is a balanced sutured manifold so that $M$ is irreducible. Then $(M, \gamma)$ is taut if and only if $\text{SH}(M, \gamma) \neq 0$. 
If \( S \subset (M, \gamma) \) is not admissible, then we can perform an isotopy on \( S \) to make it admissible.

**Definition 2.7.** Suppose \((M, \gamma)\) is a balanced sutured manifold, and \( S \) is a properly embedded surface. A stabilization of \( S \) is a surface \( S' \) obtained from \( S \) by isotopy in the following sense. This isotopy creates a new pair of intersection points:

\[
\partial S' \cap \gamma = (\partial S \cap \gamma) \cup \{p_+, p_-\}.
\]

We require that there are arcs \( \alpha \subset \partial S' \) and \( \beta \subset \gamma \), oriented in the same way as \( \partial S' \) and \( \gamma \), respectively, and the followings hold.

1. \( \partial \alpha = \partial \beta = \{p_+, p_-\} \).
2. \( \alpha \) and \( \beta \) cobound a disk \( D \) with \( \text{int}(D) \cap (\gamma \cup \partial S') = \emptyset \).

The stabilization is called **negative** if \( \partial D \) is the union of \( \alpha \) and \( \beta \) as an oriented curve. It is called **positive** if \( \partial D = (-\alpha) \cup \beta \). See Figure 1. We denote by \( S^{\pm k} \) the surface obtained from \( S \) by performing \( k \) positive or negative stabilizations, respectively.

![Figure 1. The positive and negative stabilizations of \( S \).](image)

**Remark 2.8.** The definition of stabilizations of a surface depends on the orientations of the suture and the surface. If we reverse the orientation of the suture or the surface, then positive and negative stabilizations switch between each other.

One can also relate the gradings associated to different stabilizations of a fixed surface. The results for monopole theory and instanton theory in [Li19, Wan20] can be adapted to the formal sutured homology associated to any other Floer-type theory.

**Theorem 2.9** ([Li19 Proposition 4.3] and [Wan20 Proposition 4.17]). Suppose \((M, \gamma)\) is a balanced sutured manifold and \( S \) is a properly embedded surface in \( M \), which intersects the suture \( \gamma \) transversely. Suppose \( S \) has a distinguished boundary component so that all the stabilizations mentioned below are performed on this boundary component. Then, for any \( p, k, l \in \mathbb{Z} \) so that the stabilized surfaces \( S^p \) and \( S^{p+2k} \) are both admissible, we have

\[
\text{SH}(M, \gamma, S^p, l) = \text{SH}(M, \gamma, S^{p+2k}, l + k).
\]

Note \( S^p \) is a stabilization of \( S \) as introduced in **Definition 2.7** and, in particular, \( S^0 = S \).
If we have multiple admissible surfaces, then they together induce a multi-grading. This is proved for $SHM$ and $SHI$ by Ghosh and the first author [GL19]. The proof can be adapted to our case without essential changes.

**Theorem 2.10** ([GL19 Proposition 1.14]). Suppose $(M, \gamma)$ is a balanced sutured manifold and $S_1, \ldots, S_n$ are admissible surfaces in $(M, \gamma)$. Then there exists a $\mathbb{Z}^n$-grading on $SH(M, \gamma)$ induced by $S_1, \ldots, S_n$, which we write as

$$SH(M, \gamma) = \bigoplus_{(i_1, \ldots, i_n) \in \mathbb{Z}^n} SH(M, \gamma, (S_1, \ldots, S_n), (i_1, \ldots, i_n)).$$

**Theorem 2.11** ([GL19 Theorem 1.12]). Suppose $(M, \gamma)$ is a balanced sutured manifold and $\alpha \in H_2(M, \partial M)$ is a nontrivial homology class. Suppose $S_1$ and $S_2$ are two admissible surfaces in $(M, \gamma)$ such that

$$[S_1] = [S_2] = \alpha \in H_2(M, \partial M).$$

Then, there exists a constant $C$ so that

$$SH(M, \gamma, S_1, l) = SH(M, \gamma, S_2, l + C).$$

Based on the $\mathbb{Z}^n$ grading from Theorem 2.10 we can define the graded Euler characteristic.

**Definition 2.12.** Suppose $(M, \gamma)$ is a balanced sutured manifold and $S_1, \ldots, S_n$ are admissible surfaces in $(M, \gamma)$ such that $[S_1], \ldots, [S_n]$ generate $H_2(M, \partial M)$. Let $\rho_1, \ldots, \rho_n \in H' = H_1(M)/\text{Tors}$ satisfying $\rho_i \cdot S_j = \delta_{i,j}$. The graded Euler characteristic of $SH(M, \gamma)$ is

$$\chi_{gr}(SH(M, \gamma)) := \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}^n} \chi(SH(M, \gamma, (S_1, \ldots, S_n), (i_1, \ldots, i_n))) \cdot (\rho_1^{i_1} \cdots \rho_n^{i_n}) \in \mathbb{Z}[H']/\pm H'.$$

**Remark 2.13.** By Theorem 2.11 the definition of graded Euler characteristic is independent of the choice of $S_1, \ldots, S_n$ if we regard it as an element in $\mathbb{Z}[H']/\pm H$. If the admissible surfaces $S_1, \ldots, S_n$ and a particular closure of $(M, \gamma)$ is fixed, then the ambiguity of $\pm H$ can be removed.

The following is the main result of [LY21].

**Theorem 2.14** ([LY21 Section 4]). The element $\chi_{gr}(SH(M, \gamma))$ in $\mathbb{Z}[H']/\pm H$ is independent of the choice of Floer-type theory $H$.

### 2.3. Contact handles and bypasses.

Suppose $(M, \gamma) \subset (M', \gamma')$ is a proper inclusion of balanced sutured manifolds and suppose $\xi$ is a contact structure on $M \setminus \text{int} M$ with dividing sets $\gamma' \cup (-\gamma)$. For monopole theory and instanton theory, Baldwin and Sivek [BS16a, BS16b] (see also [Li18]) constructed a contact gluing map

$$\Phi_{\xi} : SH(-M, -\gamma) \to SH(-M', -\gamma')$$

based on contact handle decompositions and the first author [Li18] showed that the map is functorial, i.e. it is independent of the contact handle decompositions and gluing two contact structures induces composite maps. This construction can be adapted to the formal sutured homology associated to any other Floer-type theory. In this subsection, we will describe the maps associated to contact 1- and 2-handle attachments, and bypass attachments (c.f. [Hon00]).

**Contact 1-handle.** Suppose $D_-$ and $D_+$ are disjoint embedded disks in $\partial M$ which each intersect $\gamma$ in a single properly embedded arc. Consider the standard contact structure $\xi_{std}$ on the 3-ball $B^3$. We glue $(D^2 \times [-1, 1], \xi_{D^2}) \cong (B^3, \xi_{std})$ to $(M, \gamma)$ by diffeomorphisms

$$D^2 \times \{-1\} \to D_-$$

and $D^2 \times \{+1\} \to D_+,$
which preserve and reverse orientations, respectively, and identify the dividing sets with the sutures. Then we round corners as shown in Figure 2 (c.f. [BS16b, Figure 2]). Let \((M_1, \gamma_1)\) be the resulting sutured manifold.

**Figure 2.** Left, the sutured manifold \((M, \gamma)\) with two points \(p\) and \(q\) on the suture. Right, the 1-handle attachment along \(p\) and \(q\).

Suppose \((Y, R)\) is a closure of \((M_1, \gamma_1)\). By [BS16b, Section 3.2], it is also a closure of \((M, \gamma)\). Define the map associated to the contact 1-handle attachment by the identify map

\[
C_{h^1} = C_{h^1, D_-, D_+} := \text{id} : \text{SH}(-M, -\gamma) \xrightarrow{=} \text{SH}(-M_1, -\gamma_1).
\]

**Contact 2-handle.** Suppose \(\mu\) is an embedded curve in \(\partial M\) which intersects \(\gamma\) in two points. Let \(A(\mu)\) be an annular neighborhood of \(\lambda\) intersecting \(\gamma\) in two cocores. We glue \((D^2 \times [-1,1], \xi_{D^2}) \cong (B^3, \xi_{\text{std}})\) to \((M, \gamma)\) by an orientation-reversing diffeomorphism

\[
\partial D^2 \times [-1, 1] \to A(\mu),
\]

which identifies positive regions with negative regions. Then we round corners as shown in Figure 3 (c.f. [BS16b, Figure 3]). Let \((M_2, \gamma_2)\) be the resulting sutured manifold.

**Figure 3.** Left, the sutured manifold \((M, \gamma)\) and the curve \(\beta \subset \partial M\) that intersects \(\gamma\) at two points. Right, the 2-handle attachment along the curve \(\mu\).

We construct the map associated to the contact 2-handle attachment as follows. Let \(\mu'\) be the knot obtained by pushing \(\mu\) into \(M\) slightly. Suppose \((N, \gamma_N)\) is the manifold obtained from \((M, \gamma)\) by a 0-surgery along \(\mu'\) with respect to the framing from \(\partial N\). By [BS16b, Section 3.3], the sutured
A manifold \((N, \gamma_N)\) can be obtained from \((M_2, \gamma_2)\) by attaching a contact 1-handle. Since \(\mu' \subset \text{int}(M)\), the construction of the closure of \((M, \gamma)\) does not affect \(\mu'\). Thus, we can construct a cobordism between closures of \((M, \gamma)\) and \((N, \gamma_N)\) by attaching a 4-dimensional 2-handle associated to the surgery on \(\mu'\). This cobordism induces a cobordism map

\[ C_{\mu'} : \text{SH}(-M, -\gamma) \to \text{SH}(-N, -\gamma_N). \]

Consider the identity map

\[ \iota : \text{SH}(-M_2, -\gamma_2) \xrightarrow{\cong} \text{SH}(-N, -\gamma_N). \]

Define the the map associated to the contact 2-handle attachment as

\[ C_{h_2} = C_{h_2, \mu} := \iota^{-1} \circ C_{\mu'} : \text{SH}(-M, -\gamma) \to \text{SH}(-M_2, -\gamma_2). \]

**Bypass attachment.** Suppose \(\alpha\) is an embedded arc in \(\partial M\) which intersects \(\gamma\) in three points. Let \(D\) be a disk neighborhood of \(\alpha\) intersecting \(\gamma\) in three arcs. There are six endpoints after cutting \(\gamma\) along \(\alpha\). We replace three arcs in \(D\) with another three arcs as shown in Figure 4. Let \((M, \gamma')\) be the resulting sutured manifold. The arc \(\alpha\) is called a **bypass arc** and this procedure is called **bypass attachment** along \(\alpha\).

![Figure 4](image)

**Figure 4.** The bypass arc and the bypass attachment, where the orientation of \(\partial M\) is pointing out.

By Ozbagci [Ozb11, Section 3], the bypass attachment can be recovered by contact handle attachments as follows. First, one can attach a contact 1-handle along two endpoints of \(\alpha\). Then one can attach a contact 2-handle along a circle that is the union of \(\alpha\) and an arc on the attached 1-handle. Topologically, the 1-handle and the 2-handle form a canceling pair, so the diffeomorphism type of the 3-manifold does not change. However, the contact structure is changed, and the suture \(\gamma\) is replaced by \(\gamma'\). We define the **bypass map** associated to the bypass attachment as

\[ \psi_{\alpha} := C_{h_2} \circ C_{h_1} : \text{SH}(-M, -\gamma) \to \text{SH}(-M, -\gamma'). \]

In [BS16a, BS18], Baldwin and Sivek proved the bypass exact triangle for sutured monopole Floer homology and sutured instanton Floer homology. Their proof can be exported to our setup.

**Theorem 2.15** ([BS16a, Theorem 5.2] and [BS18, Theorem 1.20]). Suppose \((M, \gamma_1), (M, \gamma_2), (M, \gamma_3)\) are balanced sutured manifolds such that the underlying 3-manifolds are the same, and the sutures \(\gamma_1, \gamma_2, \gamma_3\) only differ in a disk shown in Figure 5. Then there exists an exact triangle

\[
\begin{array}{ccc}
\text{SH}(-M, -\gamma_1) & \xrightarrow{\psi_1} & \text{SH}(-M, -\gamma_2) \\
\downarrow{\psi_3} & & \downarrow{\psi_2} \\
\text{SH}(-M, -\gamma_3) & & \\
\end{array}
\]
Moreover, the maps $\psi_i$ are induced by cobordisms, hence is homogeneous with respect to the relative $\mathbb{Z}_2$ grading on $\text{SH}(M, \gamma_i)$.

![Figure 5. The bypass triangle.](image)

![Figure 6. A trivial bypass.](image)

The following proposition is straightforward from the description of the bypass map.

**Proposition 2.16.** Suppose $(M, \gamma)$ is a balanced sutured manifold and $S \subset (M, \gamma)$ is an admissible surface. Suppose the disk as in Figure 5, where we perform the bypass change, is disjoint from $\partial S$. Let $\gamma_2$ and $\gamma_3$ be the resulting two sutures. Then all the maps in the bypass exact triangle (2.1) are grading preserving, i.e., for any $i \in \mathbb{Z}$, we have an exact triangle

$$
\begin{array}{c}
\text{SH}(-M, -\gamma_1, S, i) \\
\uparrow \psi_{1,i}
\end{array}
\begin{array}{c}
\text{SH}(-M, -\gamma_2, S, i) \\
\uparrow \psi_{3,i}
\end{array}
\begin{array}{c}
\text{SH}(-M, -\gamma_3, S, i) \\
\downarrow \psi_{2,i}
\end{array}
\end{array}
$$

where $\psi_{k,i}$ are the restriction of $\psi_k$ in (2.1).

A special bypass arc $\alpha_0$ is depicted in Figure 6, where the bypass attachment along $\alpha$ is called a trivial bypass (c.f. [Hon02, Section 2.3]). Attaching a trivial bypass does not change the suture on $\partial M$ and induces a product contact structure on $\partial M \times I$. The functoriality of the contact gluing maps indicates the following proposition.

**Proposition 2.17.** A trivial bypass on $(M, \gamma)$ induces an identity map on $\text{SH}(M, \gamma)$.

### 3. Decomposition Associated to Tangles

Suppose $K$ is a rationally null-homologous knot in a closed 3-manifold $Y$, i.e. $[K] = 0 \in H_1(Y; \mathbb{Q})$. Suppose $q$ is the order of $[K]$, i.e. $q$ is the smallest number satisfying $q[K] = 0 \in H_1(Y; \mathbb{Z})$. In [LY20, Section 4], we construct a decomposition

$$I^\mathbb{C}(Y) \cong \bigoplus_{i=0}^q I^\mathbb{C}(Y, i).$$

This decomposition provides a candidate for the counterpart of the torsion spin$^c$ decompositions in monopole theory and Heegaard Floer theory.

In this section, we generalize this decomposition to rationally null-homologous tangles in balanced sutured manifolds. There is no essential difference between the proofs for knots and tangles. All arguments only depend on axioms and formal properties, so we can safely consider formal sutured homology $\text{SH}$. 
3.1. Basic setups.

In this subsection, we review the construction for tangles and collect important lemmas in Section 3.2, with mild modifications.

Suppose \((M, \gamma)\) is a balanced sutured manifold. Suppose \(T = T_1 \cup \cdots \cup T_m\) is a vertical tangle in \((M, \gamma)\) (c.f. \[XZ19\] Definition 1.1), i.e., a properly embedded 1-submanifold with
\[|T_i \cap R_+ (\gamma)| = |T_i \cap R_- (\gamma)| = 1.\]

Let \(T_i\) be oriented from \(R_+ (\gamma)\) to \(R_- (\gamma)\). Throughout this subsection, we consider one component \(\alpha\) of \(T\) and assume it is rationally null-homologous, i.e., \([\alpha] = 0 \in H_1 (M, \partial M; \mathbb{Q})\). Without loss of generality, suppose \(\alpha = T_1\).

We can construct a new balanced sutured manifold \((M_T, \gamma_T)\) as follows. Let \(M_T\) be obtained from \(M\) by removing a neighborhood \(N(T) = \bigcup_{i=1}^m N(T_i)\) of \(T\). Suppose \(\gamma_i\) is a positively oriented meridian of \(T_i\) on \(\partial N(T_i)\). Define
\[\gamma_T = \gamma \cup \gamma_1 \cup \cdots \cup \gamma_m.\]

Since \(\alpha\) is rationally null-homologous, there exists a surface \(S\) in \(M\) with \(\partial S\) consisting of arcs \(\beta_1, \ldots, \beta_k\) and \(q\) copies of \(\alpha\) for some integers \(k\) and \(q\). Here \(q\) is the order of \(\alpha\).

The surface \(S\) can be modified into a properly embedded surface \(S_T\) in \(M_T\) as follows. First, for \(q\) arcs in \(\partial S\) parallel to \(\alpha\), we isotop them to be on \(\partial N(\alpha)\). Then \(\beta_1, \ldots, \beta_k\) can be regarded as arcs on \(\partial M_T\). Second, we can isotop \(S\) to make it intersect \(T_2, \ldots, T_m\) transversely. Then removing disks in \(N(T_i) \cap S\) for all \(i = 2, \ldots, m\) induces a properly embedded surface \(S_T\) in \(M_T\). Note that \(\partial S_T\) intersects \(\gamma_1\) at \(q\) points, one for each arc parallel to \(\alpha\), and the part of \(\partial S_T\) on \(\partial N(T_i)\) consists of circles parallel to \(\gamma_i\) for \(i = 2, \ldots, m\).

Suppose \(p_+\) and \(p_-\) are the endpoints of \(\alpha\) on \(R_+ (\gamma)\) and \(R_- (\gamma)\), respectively. Choose an arc \(\zeta_+ \subset R_+ (\gamma)\) connecting \(p_+\) and \(\gamma\). The arc \(\zeta_-\) induces an arc on \(R_+ (\gamma_T)\) connecting \(\gamma_1\) to \(\gamma\) such that the part on \(\partial N(\alpha)\) is parallel to \(\alpha\). We still denote this arc by \(\zeta_+\) for simplicity. Similarly we can choose an arc \(\zeta_- \subset R_+ (\gamma_T)\) connecting \(\gamma_1\) to \(\gamma\).

Let \(\Gamma_0\) be obtained from \(\gamma_T\) by band sum operations along \(\zeta_+\) and \(\zeta_-\). Then let \(\Gamma_n\) be obtained from \(\Gamma_0\) by twisting along \((-\gamma_1)\) for \(n\) times. Moreover, let \(\Gamma_+\) be the suture as depicted in Figure 7 and let \(\Gamma_- = \gamma_T\).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{The arcs \(\zeta_+, \zeta_-\), the sutures \(\Gamma_-, \Gamma_0, \Gamma_n, \Gamma_+,\) and the bypass arcs \(\eta_+, \eta_-\).}
\end{figure}

Remark 3.1. The construction of \(\zeta_+\) and \(\zeta_-\) here is a little different from the one in \[LY20\] Section 3.2, where we used \(\beta_i\) to construct \(\zeta_\alpha\) and removed a trivial tangle from \(M_T\) to obtain a manifold \(M_{T_0}\). Hence the construction of \(\Gamma_n, \Gamma_\pm\) is also different. In particular, they were on \(M_{T_0}\) in the
construction of [LY20 Section 3.2]. However, it turns out that removing the trivial tangle is not necessary and we can decompose $M_{T_0}$ along a product disk to recover $M_T$ in [LY20 Section 3.2, Step 3]. Thus, we can consider sutures on $M_T$ and all results in [LY20 Section 3.2] apply without essential change. Also, the conditions that $\zeta_\pm$ are disjoint from $\beta_1, \ldots, \beta_k$ are not essential.

There are two straightforward choices of bypass arcs on $\Gamma_n$ in Figure 7, denoted by $\eta_+$ and $\eta_-\,$, respectively. It is straightforward to check that these two bypass arcs induce the following bypass exact triangles from Theorem 2.15 (c.f. the left two subfigures of Figure 8).

\[ (3.1) \quad \text{SH}(M_T, 0) \xrightarrow{\psi^+_1, \Gamma_n} \text{SH}(M_T, 0) \xrightarrow{\psi^-_1, \Gamma_n} \text{SH}(M_T, 0) \]

The bypasses are attached along $\eta_+$ and $\eta_-$ from the exterior of the 3-manifold $M_{T_0}$, though the point of view in Figure 7 is from the interior of the manifold. So readers have to take extra care when performing these bypass attachments.

Since the bypass arcs $\eta_+$ and $\eta_-$ are disjoint from $\partial S_T$, the bypass maps in the exact triangles (3.1) preserve gradings associated to $S_T$ by Proposition 2.16. We describe it precisely as follows.

**Definition 3.2.** Suppose $(M, \gamma)$ is a balanced sutured manifold and $S$ is an admissible surface in $(M, \gamma)$. For any $i, j \in \mathbb{Z}$, define

\[ \text{SH}(M, \gamma, S, i)[j] = \text{SH}(M, \gamma, S, i-j). \]

**Lemma 3.3 ([LY20 Section 3.2]).** For any $j \in \mathbb{N} \cup \{+, -\}$, there exists admissible surfaces $S_j$ with respect to $(M_T, \Gamma_j)$ obtained from $S_T$ by stabilizations and integers $i^i_{\max}$ and $i^i_{\min}$ such that

\[ \lim_{n \to +\infty} i^n_{\max} = +\infty, \quad \lim_{n \to +\infty} i^n_{\min} = -\infty, \]

and

\[ \text{SH}(M_T, -\Gamma_j, S_j, i) = 0 \text{ for } i \notin [i^i_{\min}, i^i_{\max}]. \]
Moreover, for any \( n \in \mathbb{N} \), there are two exact triangles

\[
\begin{align*}
\text{SH}(-M_T, -\Gamma_n, S_n)[i_{\text{min}}^{n+1} - i_{\text{min}}^n] & \xrightarrow{\psi^+_{i_{\text{min}}^{n+1}}} \text{SH}(-M_T, -\Gamma_{n+1}, S_{n+1}) \\
\text{SH}(-M_T, -\Gamma_n, S_n)[i_{\text{max}}^{n+1} - i_{\text{max}}^n] & \xrightarrow{\psi^-_{i_{\text{max}}^{n+1}}} \\
\text{SH}(-M_T, -\Gamma_n, S_n)[i_{\text{min}}^{n+1} - i_{\text{min}}^n] & \xrightarrow{\psi^-_{i_{\text{min}}^{n+1}}}
\end{align*}
\]

and

\[
\begin{align*}
\text{SH}(-M_T, -\Gamma_n, S_n)[i_{\text{max}}^{n+1} - i_{\text{max}}^n] & \xrightarrow{\psi^+_{i_{\text{max}}^{n+1}}} \text{SH}(-M_T, -\Gamma_{n+1}, S_{n+1}) \\
\text{SH}(-M_T, -\Gamma_n, S_n)[i_{\text{min}}^{n+1} - i_{\text{min}}^n] & \xrightarrow{\psi^-_{i_{\text{min}}^{n+1}}}
\end{align*}
\]

Furthermore, all maps in the above two exact triangles are grading preserving.

Remark 3.4. For \( j \in \mathbb{N} \cup \{+, -\} \), the surfaces \( S_j \) and integers \( i_{\text{max}}, i_{\text{min}} \) were defined explicitly in [LY20 Section 3.2]. However, three conditions about \( S_T \) at the start of [LY20 Step 2 in Section 3.2] are not necessary. We can choose \( S_j \) to be either \( S_T \) or \( S_T^{-1} \) (the negative stabilization of \( S_T \) with respect to \( \Gamma \), c.f. Definition 2.7), which is admissible with respect to \( \Gamma_j \). The choice is denoted by \( S_j = S_T^{(j)} \) for \( \tau(j) \in \{0, -1\} \). For the definitions of \( i_{\text{max}}, i_{\text{min}} \), consider the closure \((Y_j, R_j)\) of \((M_T, \Gamma_j)\) such that \( S_j \) extends to a closed surface \( \bar{S}_j \subset Y_j \). Define

\[
i_{\text{max}} = -\frac{1}{2} \chi(\bar{S}_j), \quad \text{and} \quad i_{\text{min}} = \frac{1}{2} \chi(\bar{S}_j) - \tau(j).
\]

Moreover, we have

\[
\chi(\bar{S}_j) = \chi(S_j) - \frac{1}{2} |S_j \cap \Gamma_j|
\]

Hence

\[
\chi(\bar{S}_j) = \chi(\bar{S}_+) - q + \tau(-) \quad \text{and} \quad \chi(\bar{S}_n) = \chi(\bar{S}_+) - nq + \tau(n) \quad \text{for} \quad n \in \mathbb{N},
\]

where \( q \) is the order of the tangle \( \alpha \). The vanishing results follow from term (2) of Theorem 2.4 and a priori we do not know if \( \text{SH}(-M_T, -\Gamma_j, S_j, i) \) is non-vanishing for \( i \in \{i_{\text{min}}, i_{\text{max}}\} \).

From the vanishing results and the exact triangles in Lemma 3.3 the following lemma is straightforward. For any \( i \in \mathbb{Z}, n \in \mathbb{N} \), let \( \psi^+_{i,n+1} \) be the restriction of \( \psi^+_{i,n+1} \) on the \( i \)-th grading associated to \( S_n \).

Lemma 3.5 ([LY20 Section 3.2]). The map

\[
\psi^+_{i,n+1} : \text{SH}(-M_T, -\Gamma_n, S_n, i) \to \text{SH}(-M_T, -\Gamma_{n+1}, S_{n+1}, i + (i_{\text{min}}^{n+1} - i_{\text{min}}^n))
\]

is an isomorphism if

\[
i < P_n := i_{\text{max}}^{n+1} + (i_{\text{min}}^{n+1} - i_{\text{min}}^n) - (i_{\text{min}}^+ - i_{\text{min}}^-)
\]

\[
= i_{\text{min}}^- + (i_{\text{max}}^+ - i_{\text{min}}^+) - (i_{\text{max}}^+ - i_{\text{min}}^-)
\]

\[
= i_{\text{min}}^- + (\chi(\bar{S}_+) + (n + 1)q) - (\chi(\bar{S}_+) + \tau(\bar{S}_+))
\]

\[
= i_{\text{min}}^- + (n + 1)q - \tau(\bar{S}_+).
\]

Similarly, the map

\[
\psi^-_{i,n+1} : \text{SH}(-M_T, -\Gamma_n, S_n, i) \to \text{SH}(-M_T, -\Gamma_{n+1}, S_{n+1}, i + (i_{\text{max}}^{n+1} - i_{\text{max}}^n))
\]
is an isomorphism if

\[
i > \rho_n := i_{\min}^{n+1} - (i_{\max}^{n+1} - i_{\min}^n) + (i_{\max}^n - i_{\min}^n) = i_{\max}^n - (i_{\max}^{n+1} - i_{\min}^n) + (i_{\max}^n - i_{\min}^n) = i_{\max}^n - (-\chi(S_\gamma) + (n+1)q) + (-\chi(S_\gamma) - q) = i_{\max}^n - nq.
\]

There is another important exact triangle induced by the surgery exact triangle in Axiom (A2).

**Lemma 3.6 ([LY20], Section 3.2).** Suppose \( T' = T \setminus \alpha = T_2 \cup \cdots \cup T_m \). Then for any \( n \in \mathbb{N} \), there is an exact triangle

\[
\text{SH}(-M_T, -\Gamma_n) \xrightarrow{G_n} \text{SH}(-M_T, -\Gamma_{n+1}) \xrightarrow{F_{n+1}} \text{SH}(-M_{T'}, -\gamma_{T'})
\]

Furthermore, we have two commutative diagrams related to \( \psi^n_{+, n+1} \) and \( \psi^n_{-, n+1} \), respectively

\[
\text{SH}(-M_T, -\Gamma_n) \xrightarrow{\psi^n_{+, n+1}} \text{SH}(-M_T, -\Gamma_{n+1}) \xrightarrow{G_n} \text{SH}(-M_{T'}, -\gamma_{T'})
\]

The proof of this lemma is used in Section 4. So we sketch the proof for the reader’s convenience.

**Sketch of the proof of Lemma 3.6.** Recall that \( \gamma_1 \) is the meridian of \( \alpha \) on \( \partial M_T \). Let \( \gamma'_1 \) be the curve obtained by pushing \( \gamma_1 \) into the interior of \( M_T \), with the framing induced by \( \partial M_T \). Consider a closure \((Y, R)\) of \((-M_T, -\Gamma_{n+1})\). Consider the \((0, 1, \infty)\)-surgery triangle associated to \( \gamma'_1 \) in \( Y \). Since \( \gamma'_1 \) is in \( M_T \), the resulting closed 3-manifolds are still closures of the resulting sutured manifolds. Note that

\[
((-M_T)_{\infty}, -\Gamma_{n+1}) \cong (-M_T, -\Gamma_{n+1}), ((-M_T)_1, -\Gamma_{n+1}) \cong (-M_T, -\Gamma_n),
\]

and the 0-surgery corresponds to the map associated to the contact 2-handle attachment along \( \gamma_1 \) (c.f. Subsection 2.3). Since \((-M_T, -\gamma_{T'})\) is the sutured manifold obtained from \((-M_T, -\gamma_n)\) by attaching a contact 2-handle along \( \gamma_1 \), we obtain the desired exact triangle.

For the commutative diagram involving \( \psi^n_{+, n+1} \), note that \( \gamma'_1 \) is disjoint from the bypass arc \( \eta_+ \). Hence the related maps commute with each other:

\[
\psi^n_{+, n+1} \circ G_n = G_{n+1} \circ \psi^n_{\eta'_+},
\]

where \( \eta'_+ \) is the bypass arc as shown in the last subfigure of Figure 8. The arc \( \eta'_+ \) induces a trivial bypass, and hence by Proposition 2.17 induces an identity map on the formal sutured homology. The other commutative diagram involving \( \psi^n_{-, n+1} \) can be proved similarly.

\(\square\)
Remark 3.7. Indeed, we have two more commutative diagrams about $F_n$:

\[
\begin{array}{ccc}
\text{SH}(-M_T, -\Gamma_n) & \xrightarrow{\psi_{n+1}^n} & \text{SH}(-M_T, -\Gamma_{n+1}) \\
\downarrow{F_n} & & \downarrow{F_{n+1}} \\
\text{SH}(-M_T, -\gamma_T) & & \\
\end{array}
\]

The proofs are also similar.

Combining Lemma 3.3, Lemma 3.5, and Lemma 3.6, we get the following result.

Lemma 3.8 ([LY20, Section 3.2]). For a large enough integer $n$, the map $G_n$ in Lemma 3.6 is zero. Hence $F_{n+1}$ is surjective by the exact triangle (3.2).

The map $F_{n+1}$ in Lemma 3.6 is a map associated to a contact 2-handle attachment. We have the following grading preserving result.

Lemma 3.9 ([LY21, Section 4.2]). Suppose $(M, \gamma)$ is a balanced sutured manifold and $S \subset (M, \gamma)$ is an admissible surface. Suppose $\alpha \subset M$ is a properly embedded arc that intersects $S$ transversely and $\partial \alpha \cap \partial S = \emptyset$. Let $N = M \setminus \text{int} N(\alpha)$, $S_N = S \cap N$, and let $\mu \subset \partial N$ be the meridian of $\alpha$ which is disjoint from $\partial S_N$. Suppose $\gamma_N$ is a suture on $\partial N$ such that $(N, \gamma_N)$ is a balanced sutured manifold and attaching a contact 2-handle along $\mu$ gives $(M, \gamma)$. Let $C_h^2, \mu$ be the map associated to the contact 2-handle attachment. Then for any $i \in \mathbb{Z}$, we have

$$C_h^2, \mu(\text{SH}(-N, -\gamma_N, S_N, i)) \subset \text{SH}(-M, -\gamma, S, i).$$

3.2. One tangle component.

In this subsection, we apply lemmas in Section 3.1 to obtain a decomposition of formal sutured homology associated to one tangle component. The results in this subsection are a generalization of [LY20, Section 4.3], where we dealt with rationally null-homologous knots. The proofs are almost identical, so we omit details and only point out the difference.

We adapt the notations in Subsection 3.1. Suppose $(M, \gamma)$ is a balanced sutured manifold and suppose $T \subset (M, \gamma)$ is a vertical tangle with only one component $\alpha = T_1$, which is rationally null-homologous of order $q$. Let $M_T$ be the manifold obtained from $M$ by removing a neighborhood of $T$ and let $\gamma_T = \gamma \cup m_\alpha$, where $m_\alpha$ is a positively oriented meridian of $\alpha$.

We start with the following lemma, which roughly says the summands in the ‘middle’ gradings of $\text{SH}(-M_T, -\Gamma_n)$ associated to $S_n$ are cyclic of order $q$.

Lemma 3.10. Suppose $n \in \mathbb{N}$ and $i_1, i_2 \in \mathbb{Z}$ satisfying $i_1, i_2 \in (\rho_n, P_n)$ and $i_1 - i_2 = q$, where $\rho_n$ and $P_n$ are constants in Lemma 3.5,

$$\rho_n = i_{\max}^n - nq \quad \text{and} \quad P_n = i_{\min}^n + (n + 1)q - \tau(+)$$

Then we have

$$\text{SH}(-M_T, -\Gamma_n, S_n, i_1) \cong \text{SH}(-M_T, -\Gamma_n, S_n, i_2).$$

Proof. Based on Lemma 3.5, the proof is similar to that of [LY20, Lemma 4.22].
Let $SH$ be defined as in Definition 3.11. We have

$$\dim_{\mathbb{Z}} SH_{\alpha}(-M,-\gamma) = \dim_{\mathbb{Z}} SH(-M,-\gamma).$$

Proof. Based on Lemma 3.15, the proof is similar to that of [LY20] Lemma 4.27. Now we split the bypass exact triangle of $(-\Gamma_+, -\Gamma_n, -\Gamma_{n+1})$ into five blocks of sizes

$$q, -\chi(S_+) + 1, \chi(S_+) + (n - 1)q - 1, q, -\chi(S_+) + 1,$$

respectively, and split the bypass exact triangle of $(-\Gamma_-, -\Gamma_n, -\Gamma_{n+1})$ into five blocks of sizes

$$-\chi(S_+) + 1, q, \chi(S_+) + (n - 1)q - 1, -\chi(S_+) + 1, q,$$

respectively. Remark 3.12 ensures that the proof of [LY20] Lemma 4.27 applies verbatimly.

Remark 3.16. The essential difference for the case of tangles is that $\Gamma_+$ is not equal to $\Gamma_-$, though it is true in the case of knots in Remark 3.14.

Proposition 3.17. Suppose $n \in \mathbb{N}$ is large enough. Then the map $F_n$ in Lemma 3.6 restricted to $SH_{\alpha}(-M,-\gamma)$ is an isomorphism, i.e.

$$F_n|_{SH_{\alpha}(-M,-\gamma)} : SH_{\alpha}(-M,-\gamma) \xrightarrow{\cong} SH(-M,-\gamma).$$
Remark 3.18. In Definition 3.11, we use a large enough integer \( n \) to define \( S\mathcal{H}_\alpha(-M,-\gamma) \). We can also define \( \Gamma_{-n} \) from \( \Gamma_0 \) by twisting along \( \gamma_1 \) for \( n \) times. For a large enough integer \( n \), we can define a vector space \( S\mathcal{H}_{\alpha}'(-M,-\gamma) \) generalizing \( \mathcal{I}_-(-\hat{Y},\hat{K}) \) in \cite[Definition 4.29]{LY20}. However, from the discussion in \cite[Section 4.4]{LY20} between \( \mathcal{I}_+(-\hat{Y},\hat{K}) \) and \( \mathcal{I}_-(-\hat{Y},\hat{K}) \), we expect that \( S\mathcal{H}_\alpha'(-M,-\gamma) \) is isomorphic to \( S\mathcal{H}_\alpha(-M,-\gamma) \) up to a \( \mathbb{Z}_q \) grading shift. Hence there is no new information and we skip the discussion here.

3.3. More tangle components.

In this subsection, we obtain a decomposition of formal sutured homology associated to more tangle components. Suppose \((M,\gamma)\) is a balanced sutured manifold. For a vertical tangle \( T \) in \( M \), let \( M_T = M \setminus \text{int} N(T) \) and let \( \gamma_T \) be the union of \( \gamma \) and positively oriented meridians of components of \( T \).

First, we prove some lemmas about homology groups.

Lemma 3.19. For any connected tangle \( \alpha \) in \( M \), we have

\[
\text{rk}_2 H_1(M_\alpha) = \begin{cases} 
\text{rk}_2 H_1(M) & \text{if } [\alpha] \neq 0 \in H_1(M,\partial M;\mathbb{Q}), \\
\text{rk}_2 H_1(M) + 1 & \text{if } [\alpha] = 0 \in H_1(M,\partial M;\mathbb{Q}).
\end{cases}
\]

(3.4)

Proof. Consider the long exact sequence associated to the pair \((M,M_\alpha)\):

\[
H^1(M,\alpha) \overset{\eta_\alpha}{\longrightarrow} H^1(\alpha) \overset{\delta_\alpha}{\longrightarrow} H^1(\alpha) \overset{\delta_\alpha}{\longrightarrow} H^2(M,\alpha) \overset{\eta_\alpha}{\longrightarrow} H^2(M) \overset{\delta_\alpha}{\longrightarrow} H^3(M,\alpha).
\]

By the excision theorem, we have

\[
H^*(M,\alpha) \cong H^*(N(\alpha),\partial N(\alpha) \cap M_\alpha) \cong H^j(D^2,\partial D^2) \cong \begin{cases} 
\mathbb{Z} & j = 2, \\
0 & j = 1, 3.
\end{cases}
\]

Since \( H^2(N(\alpha),\partial N(\alpha) \cap M_\alpha) \) is generated by the disk that is the Poincaré dual of \([\alpha \cap N(\alpha)]\) and \( p_2^\alpha \) is induced by the projection, the image of \( p_2^\alpha \) is generated by the Poincaré dual of \([\alpha]\). Since \( H^1(M) \) and \( H_1(M) \) always have the same rank, we obtain the rank equation from (3.4).

Lemma 3.20. Suppose \((M,\gamma)\) is a balanced sutured manifold. There exists a (possibly empty) tangle \( T = T_1 \cup \cdots \cup T_m \) in \( M \), such that \( \text{Tors}H_1(M_T) = 0 \) and for any \( T' \subset T \) and \( T_i \subset T \setminus T' \), we have

\[
[T_i] = 0 \in H_1(M_{T'},\partial M_{T'};\mathbb{Q}).
\]

(3.5)

Proof. Suppose \( \alpha \) is a connected tangle in \( M \). From (3.4) and the proof of Lemma 3.19, we have

\[
\mathbb{Z}\langle\phi_\alpha\rangle \overset{p_2^\alpha}{\longrightarrow} H^2(M) \overset{j_\alpha}{\longrightarrow} H^2(M_\alpha) \rightarrow 0,
\]

where \( \phi_\alpha \) is the Poincaré dual of \([\alpha]\). By the universal coefficient theorem, the torsion subgroups of \( H^2(M) \) and \( H_1(M) \) are isomorphic. In particular, \( \text{Tors}H^2(M) = 0 \) if and only if \( \text{Tors}H_1(M) = 0 \). Let \( \alpha \) be a rationally null-homologous tangle, then

\[
\text{Tors}H^2(M_\alpha) \cong \text{Tors}H^2(M)/\text{PD}(\alpha).
\]

Thus, we can always choose connected tangles

\[
T_1 \subset M, \ T_2 \subset M_{T_1}, \ T_3 \subset M_{T_1 \cup T_2}, \ldots, \ T_m \subset M_{T_1 \cup \cdots \cup T_{m-1}}.
\]
that are rationally null-homologous to kill the whole torsion subgroup. In other word, for \( T = T_1 \cup \cdots \cup T_m \), we have \( \text{Tors}H_1(M_T) = 0 \).

By Lemma 3.19 we have

\[
(3.6) \quad \text{rk}_ZH_1(M_T) = \text{rk}_ZH_1(M) + m.
\]

Hence for any \( T' \) and any \( T_i \) satisfy the assumption, (3.5) holds, otherwise it contradicts with the rank equality (3.6). \( \square \)

Remark 3.21. Since moving the endpoints of a tangle on the boundary of the ambient 3-manifold does not change the homology class of the tangle, we can suppose the tangle \( T \) in Lemma 3.20 is a vertical tangle. Moreover, when \( M \) has connected boundary, we can suppose endpoints of \( T \) all lie in a neighborhood of a point on the suture \( \gamma \).

Lemma 3.22. Suppose \( (M, \gamma) \) is a balanced sutured manifold and suppose \( \alpha \) is a connected rationally null-homologous tangle of order \( q \). Let \( S_\alpha \) be a Seifert surface of \( T_i \), i.e., \( \partial S_\alpha \) consists of \( q \) parallel copies of \( \alpha \) and arcs on \( \partial M \). Suppose \( S_1, \ldots, S_n \) are admissible surfaces in \( (M, \gamma) \) generating \( H_2(M, \partial M) \). Then the restriction of \( S_1, \ldots, S_n \) and \( S_\alpha \) on \( M_T \) generate \( H_2(M_T, \partial M_T) \).

Proof. From (3.4) and the proof of Lemma 3.19, we have

\[
(3.7) \quad 0 \to H^1(M) \xrightarrow{\delta^*_\alpha} H^1(M_\alpha) \xrightarrow{\delta^*_\alpha} \mathbb{Z} \langle \phi_\alpha \rangle \xrightarrow{p_2^*} H^2(M),
\]

where \( \phi_\alpha \) is the Poincaré dual of \( [\alpha] \). It is straightforward to calculate

\[
(3.8) \quad \delta^*_\alpha(PD([S_\alpha])) = q\phi_\alpha.
\]

Since \( H^1(M) \cong H_2(M, \partial M) \), we have

\[
(3.9) \quad H_2(M_\alpha, \partial M_\alpha)/H_2(M, \partial M) \cong H^1(M_\alpha)/H^1(M) \cong H^1(M_\alpha)/\text{im} \delta^*_\alpha \cong \text{im} \delta^*_\alpha \cong \ker \delta^*_\alpha.
\]

Since the image of \( p_2^* \) is the Poincaré dual of \( [\alpha] \), we have

\[
(3.10) \quad \ker p_2^* \cong \langle q\phi_\alpha \rangle.
\]

Combining (3.7), (3.8), and (3.9), we know that \( [S_\alpha] \) generates \( H_2(M_\alpha, \partial M_\alpha)/H_2(M, \partial M) \). Thus, we conclude the desired property. \( \square \)

In the rest of this subsection, we suppose \( (M, \gamma) \) is a balanced sutured manifold and \( T = T_1 \cup \cdots \cup T_m \) is a vertical tangle satisfying Lemma 3.20. Suppose the order of the first component \( T_1 \) in \( H_1(M) \) is \( q_1 \) and suppose \( S_1 \) is a Seifert surface of \( T_1 \).

Convention. We will still use \( S_1 \) to denote its restriction on \( M_{T_1} \). This also applies to other Seifert surfaces mentioned below.

We adapt the construction in Subsection 3.1. Applying results in Subsection 3.2, we have

\[
SH_{T_1}(-M, -\gamma) := \bigoplus_{i=1}^{q_1} SH(-M_{T_1}, -\Gamma_n, (S_1)_n, Q_n - i) \cong SH(-M, -\gamma),
\]

where \( n \) is a large integer, \( (S_1)_n \) is a (possibly empty) stabilization of \( S_1 \), and \( Q_n \) is a fixed integer. For simplicity, we choose a large integer \( n_1 \) such that \( (S_1)_{n_1} = S_1 \) and write

\[
\Gamma_{n_1} = \Gamma_n|_{n=n_1} \text{ and } Q_{n_1} = Q_n|_{n=n_1}.
\]
For the second component $T_2$, suppose $S_2$ is its Seifert surface in $MT_1$ with $\partial S^2$ containing $q_2$ copies of $T_2$. Now we can apply the construction in Subsection 3.1 and the results in Subsection 3.2 to $(M, \Gamma_{n_1}^1)$. For a large integer $n_2$ such that $(S_2)_{n_1} = S_2$, we define

$$\text{SH}_{T_1 \cup T_2}(-M, -\gamma) := \bigoplus_{i_1=1}^{q_1} \bigoplus_{i_2=1}^{q_2} \text{SH}(-MT_1 \cup T_2, -\Gamma_{n_2}^2, (S_1, S_2), (Q_{n_1}^1 - i_1, Q_{n_2}^2 - i_2)) \cong \text{SH}(-M, -\gamma).$$

Iterating this procedure, we have the following definition.

**Definition 3.23.** For $i = 1, \ldots, m$, suppose the component $T_k$ is rationally null-homologous of order $q_k$ in $MT_1 \cup \cdots \cup T_{k-1}$. Inductively, for $k = 1, \ldots, m$, we choose a large integer $n_k$, a suture $\Gamma_{n_k}^k \subset \partial M_{T_1 \cup \cdots \cup T_k}$, a Seifert surface $S_k = (S_k)_{n_k} \subset M_{T_1 \cup \cdots \cup T_k}$, and an integers $Q_{n_k}^k$, such that $n_k, \Gamma_{n_k}^k, S_k, Q_{n_k}^k$ depend on the choices for the first $(k - 1)$ tangles. Define

$$\text{SH}_T(-M, -\gamma) := \bigoplus_{i_1 \in [1, q_1], \ldots, i_m \in [1, q_m]} \text{SH}(-M_T, -\Gamma_{n_m}^m, (S_1, \ldots, S_m), (Q_{n_1}^1 - i_1, \ldots, Q_{n_m}^m - i_m)).$$

**Remark 3.24.** Though we only use the subscript $T$ in the notation $\text{SH}_T(-M, -\gamma)$, it is not known if $\text{SH}_T(-M, -\gamma)$ is independent of the choices of all constructions. In particular, we have to choose an order of the components to define $\text{SH}_T(-M, -\gamma)$.

Applying results in Subsection 3.2 for $m$ times, the following proposition is straightforward.

**Proposition 3.25.** $\text{SH}_T(-M, -\gamma) \cong \text{SH}(-M, -\gamma)$.

The map $H_1(M_T) \to H_1(M)$ is surjective. The $q_1$ direct summands of $\text{SH}_{T_1}(-M, -\gamma)$ correspond to the order $q_1$ torsion subgroup generated by $[T_1] \in \text{Tors}H_1(M, \partial M) \cong \text{Tors}H^2(M) \cong \text{Tors}H_2(M)$.

Hence the summands of $\text{SH}_{T_1}(-M, -\gamma)$ provide a decomposition of $\text{SH}(-M, -\gamma)$ with respect to the torsion subgroup generated by $[T_1]$. By induction and the fact that $\text{Tors}H_1(M_T) = 0$, we can regard summands in $\text{SH}_T(-M, -\gamma)$ as a decomposition of $\text{SH}(-M, -\gamma)$ with respect to $\text{Tors}H_1(M)$.

To provide a decomposition of $\text{SH}(-M, -\gamma)$ with respect to the whole $H_1(M)$ as in Theorem 1.1, we choose admissible surfaces $S_{m+1}, \ldots, S_{m+n}$ generating $H_2(M, \partial M)$. By Lemma 3.22, the restriction of $S_1, \ldots, S_{m+n}$ generate $H_2(M_T, \partial M_T)$. By Lemma 3.9, the gradings associated to these surfaces behave well under restriction.

**Definition 3.26.** Consider the construction as above. For $i = 1, \ldots, m + n$, let $\rho_1, \ldots, \rho_{m+n} \in H_1(M_T) = H_1(M_T)/\text{Tors}$ be the class satisfying $\rho_i : S_j = \delta_{i,j}$. Consider $j_* : \mathbb{Z}[H_1(M_T)] \to \mathbb{Z}[H_1(M)]$.

We write

$$H = H_1(M), S = (S_1, \ldots, S_{m+n}), -i'_k = Q_{n_k}^k - i_{n+k}$$

and

$$-i' = (-i'_1, \ldots, -i'_{m+1}, \ldots, -i'_{m+n}), \rho^{-i'} = \rho_1^{-i'_1} \cdot \rho_{m+1}^{-i'_{m+1}} \cdots \rho_{m+n}^{-i'_{m+n}}.$$

The enhanced Euler characteristic of $\text{SH}(-M, -\gamma)$ is

$$\chi_{en}(\text{SH}(-M, -\gamma)) = j_*(\chi(\text{SH}_T(-M, -\gamma))) := j_*\left( \sum_{i_1 \in [1, q_1], \ldots, i_m \in [1, q_m]} \chi(\text{SH}(-M_T, -\gamma_T, S, -i')) \cdot \rho^{-i'} \right) \in \mathbb{Z}[H]/\pm H.$$
For $h \in H_1(M)$, let $\text{SH}(-M, -\gamma, h)$ be image of the summand of $\text{SH}_T(-M, -\gamma)$ under the isomorphism in Proposition 3.25 whose corresponding element in $\chi_{\text{en}}(\text{SH}(-M, -\gamma))$ is $h$.

Remark 3.27. As mentioned in Remark 3.24 the definition of $\text{SH}(-M, -\gamma, h)$ is not canonical, i.e. it may depend on many auxiliary choices. After fixing these choices, it is still only well-defined up to a global grading shift by multiplication by an element in $h_0 \in H_1(M)$. However, by Theorem 5.1 the enhanced Euler characteristic $\chi_{\text{en}}(\text{SH}(-M, -\gamma))$ only depends on $(M, \gamma)$.

4. Sutured Heegaard Floer homology

In this section, we discuss properties of sutured (Heegaard) Floer homology $SFH$ that are similar to those for formal sutured homology, so we can apply results in Section 3 to $SFH$. Since $SFH$ is not defined by closures of sutured manifolds, the maps associated to surgeries and contact handle attachments are different from those for $\text{SH}$.

4.1. Construction and gradings.

In this subsection, we describe the definition of $SFH$ and discuss the gradings on $SFH$ associated to admissible surfaces.

Definition 4.1 ([Juha06 Section 2]). A balanced diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a tuple satisfying the following.

1. $\Sigma$ is a compact, oriented surface with boundary.
2. $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$ are two sets of pairwise disjoint simple closed curves in the interior of $\Sigma$.
3. The maps $\pi_0(\partial \Sigma) \to \pi_0(\Sigma \backslash \alpha)$ and $\pi_0(\partial \Sigma) \to \pi_0(\Sigma \backslash \beta)$ are surjective.

For such triple, let $N$ be the 3-manifold obtained from $\Sigma \times [-1, 1]$ by attaching 3-dimensional 2–handles along $\alpha_i \times \{-1\}$ and $\beta_i \times \{1\}$ for $i = 1, \ldots, n$ and let $\nu = \partial \Sigma \times \{0\}$. A balanced diagram $(\Sigma, \alpha, \beta)$ is called compatible with a balanced sutured manifold $(M, \gamma)$ if the balanced sutured manifold $(N, \nu)$ is diffeomorphic to $(M, \gamma)$.

Suppose $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a balanced diagram with $g = g(\Sigma)$ and $n = |\alpha| = |\beta|$.

Convention. In this paper, we always suppose balanced diagrams satisfy the admissible condition in [Juha06 Section 3].

Consider two tori

$$T_\alpha := \alpha_1 \times \cdots \times \alpha_n \quad \text{and} \quad T_\beta := \beta_1 \times \cdots \times \beta_n$$

in the symmetric product

$$\text{Sym}^n \Sigma := \left( \prod_{i=1}^n \Sigma \right)/S_n.$$
Based on the above construction, Juhász [Juh06] defined a differential on $SFC(H)$ by

$$\partial_J(x) = \sum_{y \in \mathbb{Z}_0 \cap T_0} \sum_{\phi \in \pi_2(x, y) \mu(\phi) = 1} \#M_J(\phi) \cdot y.$$ 

**Theorem 4.2 ([Juh06, [JTZ18]).** Suppose $(M, \gamma)$ is a balanced sutured manifold. Then there is an admissible balanced diagram $\mathcal{H}$ compatible with $(M, \gamma)$. The vector spaces $H(SFC(H), \partial_J)$ for different choices of $\mathcal{H}$ and $J$, together with some canonical maps, form a transitive system $SFH(M, \gamma)$ over $\mathbb{F}_2$.

For a balanced sutured manifold $(M, \gamma)$, we can decompose $SFH(M, \gamma)$ along spin$^c$ structures.

Fix a Riemannian metric $g$ on $M$. Let $v_0$ be a nowhere vanishing vector field along $\partial M$ that points into $M$ along $R_-(\gamma)$, points out of $M$ along $R_+(\gamma)$, and on $\gamma$ it is the gradient of the height function $A(\gamma) \times I \to I$. The space of such vector fields is contractible, so the choice of $v_0$ is not important.

Suppose $v$ and $w$ are nowhere vanishing vector fields on $M$ that agree with $v_0$ on $\partial M$. They are called homologous if there is an open ball $B \subset \text{int} M$ such that $v$ and $w$ are homotopic on $M \setminus B$ through nowhere vanishing vector fields rel $\partial M$. Let Spin$^c(M, \gamma)$ be the set of homology classes of nowhere vanishing vector fields $v$ on $M$ with $v|_{\partial M} = v_0$. Note that Spin$^c(M, \gamma)$ is an affine space over $H^2(M, \partial M)$.

Suppose $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a balanced diagram compatible with $(M, \gamma)$. For each intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we can assign a spin$^c$ structure $s(x) \in \text{Spin}^c(M, \gamma)$ as follows (c.f. [Juh06 Section 4]).

we choose a self-indexing Morse function $f : M \to [-1, 4]$ such that

$$f^{-1}(\frac{3}{2}) = \Sigma \times \{0\}.$$ 

Moreover, curves $\alpha, \beta$ are intersections of $\Sigma \times \{0\}$ with the ascending and descending manifolds of the index 1 and 2 critical points of $f$, respectively. Then any intersection point of $\alpha_i \subset \alpha$ and $\beta_j \subset \beta$ corresponds to a trajectory of grad$f$ connecting an index 1 critical point to a index 2 critical point. For $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\gamma_x$ be the multi-trajectory corresponding to intersection points in $x$.

In a neighborhood $N(\gamma_x)$, we can modify grad$f$ to obtain a nowhere vanishing vector field $v$ on $M$ such that $v|_{\partial M} = v_0$. Let $s(x) \in \text{Spin}^c(M, \gamma)$ be the homology class of this vector field $v$.

From the assignment of the spin$^c$ structure, we have the following proposition.

**Proposition 4.3.** For any $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have

$$s(x) - s(y) = \text{PD}([\gamma_x - \gamma_y]),$$ 

where PD : $H_1(M) \to H^2(M, \partial M)$ is the Poincaré duality map.

It can be shown that there is no differential between generators corresponding to different spin$^c$ structures. Hence we have the following decomposition.

**Proposition 4.4 ([Juh06].** For any balanced sutured manifold $(M, \gamma)$, there is a decomposition

$$SFH(M, \gamma) = \bigoplus_{s \in \text{Spin}^c(M, \partial M)} SFH(M, \gamma, s).$$

Suppose $S \subset (M, \gamma)$ is an admissible surface $S$. To associate a $\mathbb{Z}$-grading on $SFH(M, \gamma)$ similar to Subsection 2.2, we need to suppose $(M, \gamma)$ is strongly balanced, i.e. for every component $F$ of $\partial M$, we have

$$\chi(F \cap R_+^{\gamma}) = \chi(F \cap R_-^{\gamma}).$$
Remark 4.5. If \( \partial M \) is connected, then it is automatically strongly balanced. For any balanced sutured manifold \((M, \gamma)\), we can obtain a strongly balanced manifold \((M', \gamma')\) by attaching contact 1-handles [Juh08, Remark 3.6]. In Subsection 4.4, we will show

\[
SFH(M', \gamma') \cong SFH(M, \gamma)
\]

and this isomorphism respects spin\(^c\) structures. Hence we can always deal with a strongly balanced manifold without losing any information.

Convention. When discussing the \( \mathbb{Z} \)-grading on \( SFH(M, \gamma) \) associated to an admissible surface \( S \subset (M, \gamma) \), we always suppose \((M, \gamma)\) is strongly balanced.

The following construction is based on [Juh08 Section 3].

Let \( v_0^+ \) be the plane bundle perpendicular to \( v_0 \) under the fixing Riemannian metric \( g \). The strongly balanced condition on \((M, \gamma)\) ensures that \( v_0^+ \) is trivial (c.f. [Juh08 Proposition 3.4]). Let \( t \) be a trivialization of \( v_0^+ \). Since any spin\(^c\) structure \( s \in \text{Spin}^c(M, \gamma) \) can be represented by a nonvanishing vector field \( v \) on \( M \) with \( v|_{\partial M} = v_0 \), we can define the relative Chern class

\[
c_1(s, t) := c_1(v^+, t) \in H^2(M, \partial M)
\]

by considering the plane bundle \( v^+ \) perpendicular to \( v \).

Let \( v_S \) be the positive unit normal field of \( S \). For a generic \( S \), we can suppose \( v_S \) is nowhere parallel to \( v_0 \) along \( \partial S \). Let \( p(v_S) \) be the projection of \( v_S \) into \( v_0^+ \). Note that \( p(v_S)|_{\partial S} \) is nowhere zero. Suppose the components of \( \partial S \) are \( T_1, \ldots, T_k \), oriented by the boundary orientation.

For \( i = 1, \ldots, k \), let \( r(T_i, t) \) be the rotation number \( p(v_S)|_{T_i} \) with respect to the trivialization \( t \) as we go around \( T_i \). Moreover, define

\[
r(S, t) := \sum_{i=1}^k r(T_i, t).
\]

Suppose \( T_1, \ldots, T_k \) intersect \( \gamma \) transversely. Define

\[
c(S, t) = \chi(S) - \frac{1}{2} \mid \partial S \cap \gamma \mid - r(S, t).
\]

Remark 4.6. The original definition of \( c(S, t) \) in [Juh08 Section 3] involves the index \( I(S) \), which is equal to \( \frac{1}{2} \mid \partial S \cap \gamma \mid \) when \( T_1, \ldots, T_k \) intersect \( \gamma \) transversely (c.f. [Juh08 Lemma 3.9]).

Suppose \( t_S \) is the trivialization of \( v_0^+ \) induced by \( p(v_S)|_{\partial S} \). Then for any \( v^+ \) with \( v^+|_{\partial M} = v_0^+ \) and any trivialization \( t \) of \( v_0^+ \), we have

\[
\langle c_1(v^+, t_S) - c_1(v^+, t), [S] \rangle = r(S, t)
\]

(c.f. Proof of [Juh08 Lemma 3.10]; see also [Juh10 Lemma 3.11]).

Definition 4.7. Consider the construction as above. Define

\[
SFH(M, \gamma, S, i) := \bigoplus_{s \in \text{Spin}^c(M, \gamma)} SFH(M, \gamma, s, [c_1(s, t_S), [S]] = 2i)
\]

Remark 4.8. The minus sign of \((2i)\) is to make this definition parallel to the \( \mathbb{Z} \)-grading on \( \text{SH}(M, \gamma) \) associated to \( S \). See the proofs of the following propositions.

Proposition 4.9. The decomposition in Definition 4.7 satisfies properties in Theorem 2.2, replacing \( \text{SH} \) by \( \text{SFH} \).
Proof. Term (1) follows from the adjunction inequality in [Juh10, Theorem 2]. Note that if
\[ 2i = |\partial S \cap \gamma| - \chi(S), \]
then for \( s \) corresponds to \( SFH(M, \gamma, S, i) \), we have
\[ \langle c_1(s, t_S), [S] \rangle = \chi(S) - |\partial S \cap \gamma| = c(S, t_S), \]
where the last equality follows from (4.1) and (4.2).

Term (2) follows from [Juh08, Lemma 3.10] and (4.4).

Terms (3)-(5) follow from definitions and symmetry on balanced diagrams. \( \square \)

Proposition 4.10. Consider the stabilized surfaces \( S^p \) and \( S^{p+2k} \) in Theorem 2.10. Then for any \( l \in \mathbb{Z} \), we have
\[ SFH(M, \gamma, S^p, l) = SFH(M, \gamma, S^{p+2k}, l + k). \]

Proof. Suppose \( S^+ \) and \( S^- \) are positive and negative stabilizations of \( S \). Since the stabilization operation is local, we have the following equation by direct calculation
\[ r(S^+, t) = r(S, t) - 1 \]
for any trivialization \( t \) of \( v^1_0 \). Note that \([S^+] = [S]\). Hence for \( s \in \text{Spin}^c(M, \gamma) \) corresponds to \( SFH(M, \gamma, S, i) \), we have
\[
\langle c_1(s, t_S), [S^+] \rangle = \langle c_1(s, t_S), [S^+] \rangle + r(S^+, t_S) \\
= \langle c_1(s, t_S), [S^+] \rangle + r(S, t_S) - 1 \\
= \langle c_1(s, t_S), [S^+] \rangle - 1 \\
= \langle c_1(s, t_S), [S] \rangle - 1 \\
= -2i - 1.
\]
Applying this calculation for \( 2k \) times gives the desired result. \( \square \)

Proposition 4.11. Suppose \( S_1 \) and \( S_2 \) are two admissible surfaces in \( (M, \gamma) \) such that
\[ [S_1] = [S_2] = \alpha \in H_2(M, \partial M). \]
Then there exists a constant \( C \) so that
\[ SH(M, \gamma, S_1, l) = SH(M, \gamma, S_2, l + C). \]

Proof. This follows directly from the definition. \( \square \)

4.2. Euler characteristics.

Then we can consider the Euler characteristic of \( SFH \) with respect to \( \text{Spin}^c \) structures.

Definition 4.12. For a balanced sutured manifold \( (M, \gamma) \), let the \( \mathbb{Z}_2 \) grading of \( SFH(M, \gamma) \) be induced by the sign of intersection points of \( \mathbb{T}_\alpha \) and \( \mathbb{T}_\beta \) for some compatible diagram \( H = (\Sigma, \alpha, \beta) \) (c.f. [FJR09, Section 3.4]). Suppose \( H = H_1(M) \) and choose any \( s_0 \in \text{Spin}^c(M, \gamma) \). The Euler characteristic of \( SFH(M, \gamma) \) is
\[ \chi(SFH(M, \gamma)) := \sum_{s \in \text{Spin}^c(M, \gamma)} \chi(SFH(M, \gamma, s)) \cdot \text{PD}(h) \in \mathbb{Z}[H]/\pm H, \]
where \( \text{PD} : H^2(M, \partial M) \to H_1(M) \) is the Poincaré duality map.
Theorem 4.13 ([FJR09]). Suppose \((M, \gamma)\) is a balanced sutured manifold. Then
\[
\chi(SFH(M, \gamma)) = \tau(M, \gamma),
\]
where \(\tau(M, \gamma)\) is a (Turaev-type) torsion element computed from the map
\[
\pi_1(R_-(\gamma)) \to \pi_1(M)
\]
by Fox calculus. In particular, if \((M, \gamma) = (Y(K), \gamma_K)\) for a knot \(K\) in \(Y\), then
\[
\tau(M, \gamma) = (1 - [m])\tau(Y(K)),
\]
where \(m\) is the meridian of \(K\) and \(\tau(Y(K))\) is the Turaev torsion defined in [Tur02].

Applying the formal sutured homology to Heegaard Floer theory, we can obtain another definition of sutured Floer homology, denoted by \(\text{SHF}\). Explicitly, for a closure \((Y, R)\) of \((M, \gamma)\), define
\[
\text{SHF}(M, \gamma) := HF(Y|R) = \bigoplus_{s \in \text{Spin}^c(Y)} HF^+(Y, s)
\]
and define \(\text{SHF}\) as the corresponding transitive system.

Remark 4.14. Originally, the \((3+1)\)-TQFT functor \(HF^+\) constructed by Ozsváth and Szabó [OS06a] is only defined for connected closed 3-manifolds and connected cobordisms. Zemke [Zem19] (see also [HMZ18 Zem18]) extended \(HF^+\) to disconnected 3-manifolds and cobordisms, at the cost of decorating 3-manifolds and cobordisms with (colored) basepoints and graphs, respectively. In [LY21 Section 3], we introduced a new transitive system to get rid of basepoints and graphs. Moreover, we proved an excision theorem which is essential in the proof of the naturality of the formal sutured homology. Thus, the functor \(HF^+\) fits the axioms of a Floer-type theory, and \(\text{SHF}\) is well-defined. Note that if the coefficient field is \(\mathbb{F}_2\), a projectively transitive system is also a transitive system since \(\mathbb{F}_2\) only has one nontrivial element.

The following theorem indicates the relation between \(SFH\) and \(\text{SHF}\).

Theorem 4.15 ([Lek13 Theorem 24], see also [BS20 Theorem 3.26]). Suppose \((M, \gamma)\) is a balanced sutured manifold and \((Y, R)\) is a closure of \((M, \gamma)\). Then there exists a balanced diagram \(H = (\Sigma, \alpha, \beta)\) compatible with \((M, \gamma)\) and a singly-pointed Heegaard diagram \(H' = (\Sigma', \alpha', \beta', z)\) of \(Y\) so that the followings hold.

1. \(\Sigma\) is a submanifold of \(\Sigma'\).
2. \(\alpha\) and \(\beta\) are subsets of \(\alpha'\) and \(\beta'\), respectively.
3. Suppose \(\alpha' = \alpha \cup \alpha'^\circ\) and \(\beta' = \beta \cup \beta'^\circ\). There exists an intersection point \(x_1 \in T_{\alpha'^\circ} \cap T_{\beta'^\circ}\) so that the map
\[
f : SFC(H) \to HF^+(H'|R)
\]
\[
c \mapsto [c \times x_1, 0]
\]
is a quasi-isomorphism.

Fixing any spin\(^c\) structure \(s_0 \in \text{Spin}^c(M, \gamma)\), we can identify \(\text{Spin}^c(M, \gamma)\) with \(H^2(M, \partial M)\) and then project elements to \(H^2(M, \partial M)/\text{Tors}\). By considering the quasi-isomorphism in Theorem 4.15 with respect to spin\(^c\) structures and \(\mathbb{Z}_2\) gradings, we obtained the following result.

Corollary 4.16 ([LY21 Corollary 3.42]). Suppose \((M, \gamma)\) is a balanced sutured manifold and \(H' = H_1(M)/\text{Tors} \cong H^2(M, \partial M)/\text{Tors}\). We have
\[
SFH(M, \gamma) \cong \text{SHF}(M, \gamma)
\]
with respect to the grading associated to \( H \) and the \( \mathbb{Z}_2 \) grading, up to a global grading shift.

In particular, we have

\[
p_* (\chi(SFH(M, \gamma))) = \chi_{gr}(\text{SFH}(M, \gamma)) \in \mathbb{Z}[H']/\pm H',
\]

where \( p_* \) is induced by the projection \( H_1(M) \to H_1(M)/\text{Tors} \).

**Remark 4.17.** We can also define the graded Euler characteristic \( \chi_{gr}(SFH(M, \gamma)) \) by the way similar to Definition 4.12. This alternative definition coincides with the element \( p_*(\chi(SFH(M, \gamma))) \) in Corollary 4.16 because for any spin’ structures \( s_1, s_2 \in \text{Spin}’(M, \gamma) \) and any trivialization \( t \) of \( u_0^1 \), we have

\[
c_1(s_1, t) - c_1(s_2, t) = \pm 2(s_1 - s_2).
\]

Note that \( \chi(SFH(M, \gamma)) \) may contain more information than \( \chi_{gr}(SFH(M, \gamma)) \) if \( H_1(M) \) has torsions.

### 4.3. Surgery exact triangle.

Suppose \((M, \gamma)\) is a balanced sutured manifold and \( K \) is a knot in \( M \). Consider three balanced sutured manifolds \((M_i, \gamma_i)\) for \( i = 1, 2, 3 \) obtained from \((M, \gamma)\) by Dehn surgeries along \( K \). If the Dehn filling curves \( \eta_1, \eta_2, \eta_3 \subset \partial(M - \text{int} \partial N(K)) \) satisfy

\[
\eta_1 \cdot \eta_2 = \eta_2 \cdot \eta_3 = \eta_3 \cdot \eta_1 = -1,
\]

then we have the following exact triangle of formal sutured homology from the surgery exact triangle (Axiom (A2)) in the closure of \((M_i, \gamma_i)\)

\[
\begin{array}{ccc}
\text{SFH} & \overset{1}{\longrightarrow} & \text{SFH} \\
\downarrow & & \downarrow \\
\text{SFH} & \overset{3}{\longrightarrow} & \text{SFH}
\end{array}
\]

In this subsection, we show the exact triangle (4.5) is also true when replacing \( \text{SFH} \) by \( SFH \).

First, we quickly review Juhasz’s construction of the cobordism map associated to a Dehn surgery (c.f. Juh16 Section 6), see also OS06a for Dehn surgeries on closed 3-manifolds.

For simplicity, suppose \( \eta_i \) is the meridian of \( K \). Choose an arc \( a \) connecting \( K \) to \( R_+(\gamma) \). We can construct a sutured triple diagram \((\Sigma, \alpha, \beta, \delta)\) satisfying the following properties.

1. \(|\alpha| = |\beta| = |\gamma| = \delta = d.\)
2. \((\Sigma, \alpha, \{\beta_2, \ldots, \beta_d\})\) is a diagram of \((M', \gamma') = (M \setminus N(K \cup a), \gamma)\).
3. \(\beta_2, \ldots, \beta_d\) are obtained from \(\beta_2, \ldots, \beta_d\) by small isotopy, respectively.
4. After compressing \(\Sigma\) along \(\beta_2, \ldots, \beta_d\), the induced curves \(\beta_1\) and \(\delta_1\) lie in the punctured torus \(\partial N(K) \setminus N(a)\).
5. \(\beta_1\) represents the meridian \(\eta_1\) of \(K\) and \(\delta_1\) represents the curve \(\eta_2\). In particular, \(\beta_1\) intersects \(\delta_1\) transversely at one point.

Then we can construct a 4-manifold \(\mathcal{W}_{\alpha, \beta, \delta}\) associated to \((\Sigma, \alpha, \beta, \delta)\) such that it is a cobordism from \((M, \gamma) = (M_1, \gamma_1)\) to

\[
(M_2, \gamma_2) \sqcup (R_+ \times I \times \partial R_+ \times I) #^{d-n} (S^2 \times S^1),
\]

where \(R_+ = R_+(\gamma)\) and different copies of \(S^2 \times S^1\) might be summed along different components of \(R_+ \times I\).

Choose a top dimensional generator \(\Theta_{\beta, \delta}\) of

\[
SFH(R_+ \times I \times \partial R_+ \times I) #^{d-n} (S^2 \times S^1) \cong \Lambda^* H^1(\#^{d-n} (S^2 \times S^1)).
\]
Note that \( p_\Sigma, \alpha, \beta \) is a balanced diagram of \( (M_1, \gamma_1) \) and \( p_\Sigma, \alpha, \delta \) is a balanced diagram of \( (M_2, \gamma_2) \). There is a map
\[
F_{\alpha, \beta, \gamma} : SFH(p_\Sigma, \alpha, \beta) \otimes SFH(p_\Sigma, \beta, \delta) \to SFH(p_\Sigma, \alpha, \delta)
\]
obtained by counting holomorphic triangles in \( (p_\Sigma, \alpha, \beta, \delta) \). Then define the cobordism map as
\[
F_1 : SFH(M_1, \gamma_1) \to SFH(M_2, \gamma_2)
\]
\[
F_1(x) = F_{\alpha, \beta, \gamma}(x, \Theta_{\beta, \delta})
\]
Similarly, we can define the cobordism maps \( F_2 \) and \( F_3 \).

**Theorem 4.18 (Surgery exact triangle).** Consider \( (M_i, \gamma_i) \) and cobordism maps \( F_i \) for \( i = 1, 2, 3 \) as above. Then we have an exact triangle

\[
\begin{array}{ccc}
SFH(M_1, \gamma_1) & \xrightarrow{F_1} & SFH(M_2, \gamma_2) \\
\downarrow{F_3} & & \downarrow{F_2} \\
SFH(M_3, \gamma_3) & \xleftarrow{F_2} & SFH(M_2, \gamma_2)
\end{array}
\]

**Proof.** The proof follows the proof of [OS04b, Theorem 9.12] without essential changes (see also [OS05, OS06b]). Since the cobordism maps \( F_i \) are well-defined on \( SFH \), we can verify the exact triangle for any diagram. We can construct a diagram \( p_\Sigma, \alpha, \beta, \delta, \zeta \) such that \( p_\Sigma, \alpha, \beta, \delta \) defines \( F_1 \), \( p_\Sigma, \alpha, \delta, \zeta \) defines \( F_2 \), and \( p_\Sigma, \alpha, \zeta, \beta \) defines \( F_3 \). Then we can verify the assumptions of the triangle detection lemma [OS05, Lemma 4.2] by counting holomorphic squares and pentagons and then this lemma induces the desired exact triangle. \( \square \)

### 4.4. Contact handles and bypasses.

Suppose \( (M, \gamma) \subset (M', \gamma') \) is a proper inclusion of balanced sutured manifolds and suppose \( \xi \) is a contact structure on \( M' \setminus \text{int} M \) with dividing sets \( \gamma' \cup (-\gamma) \). Honda, Kazez, and Matić [HKM08] defined a map
\[
\Phi_\xi : SFH(M, \gamma) \to SFH(M', \gamma'),
\]
which is indeed the motivation of Baldwin and Sivek’s construction in Subsection 2.3.

Originally, this map is defined by partial open book decompositions, and there are some technical conditions. Juhasz and Zemke [JZ20] provided an alternative description of this map by contact handle decompositions. Their description is explicit on balanced diagrams of sutured manifolds. We will follow this alternative definition and describe the maps for contact 1- and 2-handle attachments.

It is also worth mentioning that Zarev [Zar10] defined a gluing operation for sutured manifolds and conjectured the map associated to contact structures above can be recovered by the gluing operation. This was proved by Leigon and Salmoiraghi [LS20].

Juhasz and Zemke’s construction can be shown in Figure 9 and Figure 10 ([JZ20, Figure 1.1]). Note that for all maps associated to contact structures, we should reverse the orientations of the manifold and the suture.

Let \( (\Sigma, \alpha, \beta) \) be a balanced diagram compatible with \( (M, \gamma) \). Then \( (-\Sigma, \alpha, \beta) \) is a balanced diagram compatible with \( (-M, -\gamma) \). Attaching a 3-dimensional contact 1-handle along \( D_+ \) and \( D_- \) corresponds to attaching a 2-dimensional 1-handle along \( D_+ \cap \gamma \) and \( D_- \cap \gamma \) in \( \partial \Sigma \). This operation does not change the sutured Floer chain complex and we define \( C_h^1 = C^1_{h^1, D_+, D_-} \) as the tautological map on intersection points.
For a contact 2-handle attachment along $\mu \subset \partial M$, note that $|\mu \cap \gamma| = 2$. Suppose $\lambda_+$ and $\lambda_-$ are arcs corresponding to $\mu \cap R_+(\gamma)$ and $\mu \cap R_-(\gamma)$, respectively. After isotopy, we can suppose $\lambda_+$ and $\lambda_-$ are properly embedded arcs on $\Sigma$. We glue a 2-dimensional 1-handle $h$ along $\partial \Sigma$ to obtain $\Sigma'$, and construct two curves $\alpha_0$ and $\beta_0$ that intersect at one point $c$ in $H$, and such that

$$\alpha_0 \cap \Sigma = \lambda_+, \beta_0 \cap \Sigma = \lambda_-.$$  

Consider the balanced diagram $(\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})$ and define the map associated to the contact 2-handle attachment as

$$C_{h^2}(x) = C_{h^2, \mu}(x) := x \times \{c\}$$

for any $x \in T_\alpha \cap T_\beta$.

Since a bypass attachment can be regarded as a composition of a contact 1-handle and 2-handle attachment (c.f. Subsection 2.3), we can define the bypass map by $C_{h^2} \circ C_{h_1}$.

Honda [Hon] proposed an exact triangle associated to bypass maps for $SFH$, which is indeed the motivation of the bypass exact triangle in Theorem 2.15. A proof of the exact triangle based on bordered sutured Floer homology was provided by Etnyre, Vela-Vick, and Zarev [EVVZ17].

**Theorem 4.19** (Bypass exact triangle, [EVVZ17, Section 6]). Suppose $(M, \gamma_1)$, $(M, \gamma_2)$, $(M, \gamma_3)$ are balanced sutured manifolds such that the underlying 3-manifolds are the same, and the sutures $\gamma_1$, $\gamma_2$, and $\gamma_3$ only differ in a disk shown in Figure 5. Then there exists an exact triangle

$$SFH(-M, -\gamma_1) \xrightarrow{\psi_1} SFH(-M, -\gamma_2) \xrightarrow{\psi_2} SFH(-M, -\gamma_3)$$

where $\psi_1, \psi_2, \psi_3$ are bypass maps associated to the corresponding bypass arcs.
From Juhász and Zenke’s description of contact gluing maps, it is obvious that the maps respect the decomposition of $SFH$ by spin$^c$ structures. We describe this fact explicitly as follows.

**Lemma 4.20.** Suppose $(M, \gamma)$ is a balanced sutured manifold and suppose $(M', \gamma')$ is the resulting sutured manifold after either a contact 1-handle or 2-handle attachment. For any spin$^c$ structure $s \in \text{Spin}^c(-M, -\gamma)$, suppose $s' \in \text{Spin}^c(-M', -\gamma')$ is its extension corresponding to handle attachments. Then we have

$$C_{h^i}((SFH(-M, -\gamma, s)) \subset SFH(-M', -\gamma', s'),$$

where $i \in \{1, 2\}$.

**Proof.** We prove the claim on the chain level. After fixing a spin$^c$ structure $s_0$ on $(M, \gamma)$, we can identify $\text{Spin}^c(M, \gamma)$ with $H^2(M, \partial M) \cong H_1(M)$. Moreover, we can represent the difference of two spin$^c$ structures by a one-cycle in Proposition 4.3.

We can extend $s_0$ to a spin$^c$ structure $s_0'$ on $(M, \gamma)$ and identify $\text{Spin}^c(M', \gamma')$ with $H_1(M')$. The inclusion $i : M \to M'$ induces a map

$$i_* : H_1(M) \to H_1(M').$$

For any $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the one cycle $\gamma_x - \gamma_y$ defined in Proposition 4.3 lies in the interior of $M$. For a contact 1-handle, since the associated map $C_{h^1}$ is tautological on intersection points, the homology class $i_*([\gamma_x - \gamma_y])$ characterizes the difference of spin$^c$ structures on $(M', \gamma')$ for $x$ and $y$. For a contact 2-handle, since $\gamma_x \times \{c\}$ is the union of multi-trajectory $\gamma_x$ and the trajectory associated to $c$, we have

$$[\gamma_x \times \{c\} - \gamma_y \times \{c\}] = i_*([\gamma_x - \gamma_y]).$$

This implies the desired proposition. \qed

**Remark 4.21.** The reader can compare Lemma 4.20 with Lemma 3.9. Note that when $H_1(M)$ has torsions, preserving the spin$^c$ structures is stronger than preserving the gradings associated to an admissible surface.

**Corollary 4.22.** Suppose $\alpha$ is a bypass arc on a balanced sutured manifold $(M, \gamma)$. Suppose $(M, \gamma')$ is the resulting manifold after the bypass attachment along $\alpha$. Then the bypass map $\psi_\alpha$ for $SFH$ respects spin$^c$ structures, i.e., for any $s \in \text{Spin}^c(M, \gamma)$ and its extension $s' \in \text{Spin}^c(M, \gamma')$, we have

$$\psi_{\alpha}(SFH(-M, -\gamma, s)) \subset SFH(-M', -\gamma', s').$$

**Proof.** This follows directly from Lemma 4.20 by the fact that a bypass attachment is a composition of a contact 1-handle and 2-handle attachment. \qed

**Remark 4.23.** By Corollary 4.22 if we consider the $\mathbb{Z}$-grading associated to an admissible surface $S$ in Subsection 4.7 then the bypass exact triangle in Theorem 4.19 satisfies the similar grading shifting property to that in Lemma 5.3.

For formal sutured homology, the map associated to a contact 2-handle is defined by the composition of the inverse of a contact 1-handle map and the cobordism map of a 0-surgery. The following proposition shows that we can define the map $C_{h^2}$ for $SFH$ in the same way.

**Lemma 4.24** ([GZ]). Suppose $(M, \gamma)$ is a balanced sutured manifold and $(M', \gamma')$ is the resulting sutured manifold after a contact 2-handle attachment along $\mu \subset \partial M$. Let $\mu'$ be the framed knot obtained by pushing $\mu$ into the interior of $M$ slightly, with the framing induced from $\partial M$. Suppose $(N, \gamma_N)$ is the sutured manifold obtained from $(M, \gamma)$ by a 0-surgery along $\mu'$. Let

$$F_{\mu'} : SFH(-M, -\gamma) \to SFH(-N, -\gamma_N)$$
be the associated map. Let $D \subset N$ be the product disk which is the union of the annulus bounded by $\mu \cup \mu'$ and the meridian disk of the filling solid torus. Let 

$$C_D : SFH(-N, -\gamma_N) \to SFH(-M', -\gamma')$$ 

be the map associated to the decomposition along $D$ (i.e. the inverse of a contact 1-handle map). Then we have 

$$C_{h^2, \mu} = C_D \circ F_{\mu'} : SFH(-M, -\gamma) \to SFH(-M', -\gamma').$$

Proof. Since all maps are well-defined on $SFH$, we can verify the claim by any diagram. Suppose $(\Sigma, \alpha, \beta)$ is a balanced diagram compatible with $(M, \gamma)$. We note that the map associated to the 0-surgery along $\mu'$ may be achieved by first performing a compound stabilization and then computing a triangle map. The resulting diagram leaves an extra band which is deleted by $C_D$. By [OS04c, Theorem 9.4], the claim then follows from a model computation in the stabilization region, as shown in Figure 11.

Proof. Since all maps are well-defined on $SFH$, we can verify the claim by any diagram. Suppose $(\Sigma, \alpha, \beta)$ is a balanced diagram compatible with $(M, \gamma)$. We note that the map associated to the 0-surgery along $\mu'$ may be achieved by first performing a compound stabilization and then computing a triangle map. The resulting diagram leaves an extra band which is deleted by $C_D$. By [OS04c, Theorem 9.4], the claim then follows from a model computation in the stabilization region, as shown in Figure 11.

\[\text{Figure 11. Realizing the contact 2-handle map (bottom-most long arrow) as a composition of a compound stabilization (top), followed by a 4-dimensional 2-handle map (middle left), followed by a product disk map (middle right). A holomorphic triangle of the 2-handle map is indicated in the top subfigure.}\]

Remark 4.25. We expect the map $C_{h^2, \mu}$ on $SFH$ is equivalent to the one on $SHF$ under the isomorphism $SFH \cong SHF$ induced by Theorem 4.15. Indeed, we can embed $\Sigma$ and the stabilization region in Figure 11 into the closed surface $\Sigma'$ in the statement of Theorem 4.15. Then we only need to show the holomorphic triangles in $\Sigma'$ all lie in the stabilization region. This is a generalization of [BS20, Theorem 3.29], where $(M, \gamma)$ is a sutured handlebody $H(S)$. However, the proof of [BS20, Theorem 3.29] is technical, and we need to choose a good diagram such that $g(\Sigma')$ is large enough.
and α curves have sufficient winding. So we leave the proof of the equivalence to the future. That is also the reason why we choose to recover all results in Subsection 3 for SFH. Moreover, we also expect the equivalence of contact gluing maps for other contact handles and even the equivalence of TQFTs of SFH and SHF.

Combining the surgery exact triangle in Theorem 4.18 with Lemma 4.24, we obtain similar results in Lemma 3.6 for SFH.

**Proposition 4.26.** Consider the setups in Subsection 3.1. Suppose $T^1 = T \setminus \alpha = T_2 \cup \cdots \cup T_m$. Then for any $n \in \mathbb{N}$, there is an exact triangle

$$\text{SFH}(\mathbb{M}, \mathbb{\gamma}) \xrightarrow{F_n} \text{SFH}(\mathbb{M}, \mathbb{\gamma}) \xrightarrow{\psi_{n+1}} \text{SFH}(\mathbb{M}, \mathbb{\gamma})$$

The map $F_{n+1}$ is induced by the contact 2-handle attachment along the meridian of $\alpha$. Furthermore, we have commutative diagrams related to $\psi_{n+1}$ and $\psi_{n+1}$, respectively

$$\text{SFH}(\mathbb{M}, \mathbb{\gamma}) \xrightarrow{F_n} \text{SFH}(\mathbb{M}, \mathbb{\gamma}) \xrightarrow{\psi_{n+1}} \text{SFH}(\mathbb{M}, \mathbb{\gamma})$$

and

$$\text{SFH}(\mathbb{M}, \mathbb{\gamma}) \xrightarrow{F_{n+1}} \text{SFH}(\mathbb{M}, \mathbb{\gamma}) \xrightarrow{\psi_{n+1}} \text{SFH}(\mathbb{M}, \mathbb{\gamma})$$

**Proof.** It follows from the proof of Lemma 3.6. \qed

5. **Proof of main theorems**

In this section, we prove Theorem 1.1 and Theorem 1.3 in the introduction.

**Theorem 5.1** (Theorem 1.1). Suppose $(\mathbb{M}, \mathbb{\gamma})$ is a balanced sutured manifold. Suppose $H = H_1(\mathbb{M})$ and consider the (Turaev-type) torsion element $\tau(\mathbb{M}, \mathbb{\gamma})$ in Theorem 4.13. Then we have

$$\chi_{\text{en}}(\text{SH}(-\mathbb{M}, -\mathbb{\gamma})) = \chi(\text{SFH}(-\mathbb{M}, -\mathbb{\gamma})) = \tau(-\mathbb{M}, -\mathbb{\gamma}) \in \mathbb{Z}[H]/\pm H.$$

**Proof.** By Theorem 4.13 it suffices to prove the first equation.

First, we consider the case that $(\mathbb{M}, \mathbb{\gamma})$ is strongly balanced. By discussion in Subsection 4.1, we can construct a $\mathbb{Z}$-grading on SFH associated to an admissible surface $S \subset (\mathbb{M}, \mathbb{\gamma})$. By discussion in Section 3, this $\mathbb{Z}$-grading also satisfies properties in Subsection 3.1 for the $\mathbb{Z}$-grading on SH associated to $S$. Hence for a vertical tangle $T$ satisfies the conditions in Definition 3.23, we can define a vector space $\text{SFH}_T(-\mathbb{M}, -\mathbb{\gamma})$ similar to $\text{SH}_T(-\mathbb{M}, -\mathbb{\gamma})$ in Definition 3.23. Similar to Proposition 3.25, there is an isomorphism

$$\text{SFH}_T(-\mathbb{M}, -\mathbb{\gamma}) \cong \text{SFH}(-\mathbb{M}, -\mathbb{\gamma}).$$
Moreover, by the proofs of Lemma 3.6 and Proposition 3.17, the isomorphism in (5.1) is induced by contact 2-handle attachments along meridians of tangle components of $T$. Hence by Lemma 4.20, the isomorphism in (5.1) respects spin$^c$ structures. This implies that there exists $s_0 \in \text{Spin}^c(-M, -\gamma)$, such that for any $h \in H_1(M)$, the summand of $SFH_T(-M, -\gamma)$ corresponding to $h$ is isomorphic to $SFH(-M, -\gamma, s_0 + h)$. In particular, we have

$$\chi_{\text{en}}(SFH(-M, -\gamma)) := j_*(\chi(SFH_T(-M, -\gamma)) = \chi(SFH(-M, -\gamma)) \in \mathbb{Z}[H]/ \pm H$$

where $j_* : \mathbb{Z}[H_1(M)] \to \mathbb{Z}[H_1(M)]$.

By definition, the vector spaces $SFH_T(-M, -\gamma)$ and $SH_T(-M, -\gamma)$ are direct summands of $SFH(-M_T, \Gamma)$ and $SH(-M_T, \Gamma)$ for some $\Gamma \subseteq \partial M_T$, respectively. By Lemma 3.20, the group $H_1(M_T)$ has no torsion. Hence by Theorem 2.14, Corollary 4.16 and Remark 4.17, we have

$$\chi(SH_T(-M, -\gamma)) = \chi(SFH_T(-M, -\gamma)) \in \mathbb{Z}[H_1(M_T)]/ \pm H_1(M_T).$$

Thus, we have

$$\chi_{\text{en}}(SH(-M, -\gamma)) = \chi_{\text{en}}(SFH(-M, -\gamma)) = \chi(SFH(-M, -\gamma)) \in \mathbb{Z}[H]/ \pm H$$

Then we consider the case that $(M, \gamma)$ is not strongly balanced. As mentioned in Remark 4.5. If $\partial M$ is not connected, we can construct a sutured manifold $(M', \gamma')$ with connected boundary by attaching contact 1-handles (c.f. [Ihh05, Remark 3.6]). The product disks in $(M', \gamma')$ corresponding to these 1-handles are admissible surfaces, and only one summand in the associated $\mathbb{Z}$-grading is nontrivial. Hence there is a canonical way to consider $\chi_{\text{en}}(SH(-M', -\gamma'))$ as an element in $\mathbb{Z}[H_1(M)]/ \pm H_1(M)$. We can consider $(-M', -\gamma')$ instead, and the above arguments about strongly balanced sutured manifolds apply to this case.

**Proof of Theorem 1.3.** We prove the theorem for $(-M, -\gamma)$, and it suffices to prove that the case that $(M, \gamma)$ is strongly balanced. Consider the construction of $SFH_T(-M, -\gamma)$ in the proof of Theorem 5.1. Also, we apply the formal sutured homology to monopole theory to obtain $SHM_T(-M, -\gamma)$ and use it to provide a decomposition of $SHM(-M, -\gamma)$.

By definition, the vector spaces $SFH_T(-M, -\gamma)$ and $SHM_T(-M, -\gamma)$ are direct summands of $SFH(-M_T, \Gamma)$ and $SHM(-M_T, \Gamma)$ for some $\Gamma \subseteq \partial M_T$. By Lemma 3.20, the group $H_1(M_T)$ has no torsion. Hence it suffices to prove the theorem for $(-M_T, \Gamma)$.

We have

$$SHM(-M_T, -\Gamma) \cong SHM(-M_T, -\Gamma) \otimes \Lambda \cong SHF(-M_T, -\Gamma) \otimes \Lambda \cong SFH(-M_T, -\Gamma) \otimes \Lambda$$

with respect to the $H_1(M_T)$-grading induced by spin$^c$ structures or admissible surfaces, where $\Lambda$ is the Novikov ring over $\mathbb{Z}_2$. The first isomorphism comes from the construction of $SHM$ and $SHM$. The second isomorphism comes from the fact that the formal sutured homology is defined via closures and the isomorphism between $\check{HM}$, and $HF^+$ for closed 3-manifolds (c.f. [KLT10, Tau10, CGH17]). The third isomorphism follows from Corollary 4.16 (c.f. [Lek13, BS20]).

**6. Knots with small instanton knot homology**

In this section, we prove detection results about $\check{HFK}$ and $KHI$ for null-homologous knots inside L-spaces.
6.1. Restrictions on Euler characteristics.

In this subsection, we provide restrictions on Euler characteristics of $\overline{HFK}$ and $KHI$.

Suppose $(M, \gamma)$ is a balanced sutured manifold and $H = H_1(M; \mathbb{Z})$. In Definition 4.12, the Euler characteristic $\chi(SFH(M, \gamma))$ has an ambiguity of $\pm H$. When $M = Y(K)$ for a knot $K$ in a rational homology sphere $Y$, we have $\partial M \cong T^2$. We can resolve the ambiguity of $\pm H$ as follows.

First, we resolve the sign ambiguity. Under the map $\mathbb{Z}[H] \to \mathbb{Z}[H_1(Y)]$ induced by inclusion, the Euler characteristic $\chi(SFH(M, \gamma))$ becomes

$$\pm \sum_{h \in H_1(Y)} h$$

by Theorem 4.13. Hence we can fix the sign by choosing the one whose image is the positive one.

Then we resolve the ambiguity of $H$. We write Spin$^c(Y, K)$ for Spin$^c(Y(K), \gamma_K)$. Since $\partial Y(K) \cong T^2$, the suture $\gamma_K$ is isotopic to the suture $-\gamma_K$, which corresponds to the knot $-K$ with reverse orientation. Hence we have an involution $\iota$ on $\overline{HFK}(Y, K)$. If there is a spin$^c$ structure $s \in$ Spin$^c(Y, K)$ so that $\overline{HFK}(Y, K, s)$ that is invariant under the involution, then choose $s_0$ in Definition 4.12, so that

$$PD(s - s_0) = e,$$

where $e \in H$ is the identify element. If there is no summand that is invariant under the involution, and suppose

$$\iota(\overline{HFK}(Y, K, s)) = \overline{HFK}(Y, K, s')$$

for some $s, s' \in$ Spin$^c(Y, K)$, then we define $\chi(\overline{HFK}(Y, K))$ as an element in $(\frac{1}{2}\mathbb{Z})[H]$ so that the group elements corresponding to $\overline{HFK}(Y, K, s)$ and $\overline{HFK}(Y, K, s')$ are inverse elements. In particular, if $H \cong \mathbb{Z}$, then $(\frac{1}{2}\mathbb{Z})[H] \cong \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. Note that this definition is independent of the choice of $s, s'$.

**Definition 6.1.** Suppose $K$ is a knot in a rational homology sphere $Y$ and Suppose $H = H_1(Y(K))$. When fixing the $\mathbb{Z}_2$-grading and the spin$^c$ grading as above, the space $\overline{HFK}(Y, K)$ is called the canonical representative. The corresponding spin$^c$ grading is called the absolute Alexander grading. For the canonical representative of $\overline{HFK}(Y, K)$, the Euler characteristic $\chi(\overline{HFK}(Y, K))$ is a well-defined element in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$. For $s \in$ Spin$^c(Y, K)$, define $\overline{HFK}(Y, K, [s])$ as the direct summand of all $\overline{HFK}(Y, K, s')$ with $s' \in$ Spin$^c(Y, K)$ and $s'$ extends to $s$ on $Y$. Then $\chi(\overline{HFK}(Y, K, s))$ is also a well-defined element in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$.

Then we can state the main theorem of this subsection.

**Theorem 6.2.** Suppose $K_1$ and $K_2$ be two knots in a rational homology sphere $Y$ with $[K_1] = [K_2] \in H_1(Y)$. For $i = 1, 2$, suppose $m_i$ is the meridian of $K_i$. Then there exists an isomorphism $\phi : H_1(Y(K_i); \mathbb{Z}) \cong H_1(Y(K_2); \mathbb{Z})$ so that $\phi([m_1]) = [m_2]$. Using the isomorphism $\phi$, we write $H_1(Y(K_i); \mathbb{Z})$ as $H$ and write $[m_1]$ as $[m] \in H$.

Consider the canonical representative of $\overline{HFK}(Y, K_i)$. Then for any $s \in$ Spin$^c(Y)$, there exists a Laurent polynomial $f_s(x) \in \mathbb{Z}[x, x^{-1}]$ and an element $h_s \in H$ such that

$$\chi(\overline{HFK}(Y, K_1, [s])) - \chi(\overline{HFK}(Y, K_2, [s])) = ([m] - 1)^2 f_s([m]) h_s,$$

where both sides are elements in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$. 
Note that Theorem 6.2 is a generalization of [Ye21, Theorem 5.8]. Indeed, the proof of [Ye21, Theorem 5.8] applies without essential change. In the following, we prove generalizations of lemmas in [Ye21, Section 5] and then sketch the proof of Theorem 6.2.

Lemma 6.3. Suppose $K_1$ and $K_2$ be two knots in a rational homology sphere $Y$ with $[K_1] = [K_2] \in H_1(Y)$. Then there exists an isomorphism $\phi : H_1(Y(K_1);\mathbb{Z}) \cong H_1(Y(K_2);\mathbb{Z})$ so that $\phi([m_1]) = [m_2]$.

Proof. For a manifold $M$, let $T_1(M)$ be the torsion subgroup of $H_1(M)$ and let $B_1(M) = H_1(M)/T_1(M)$. By [Bro60, Theorem 3.1], since $[K_1] = [K_2] \in H_1(Y)$, there is a subgroup $B$ of $B_1(Y(K_1 \cup K_2))$ so that for $i = 1, 2$, the map

$$j_i : B_1(Y(K_1 \cup K_2)) \to B_1(Y(K_i))$$

induced by the injection is an isomorphism on $B$. Moreover, the map

$$k_i : T_1(Y(K_1 \cup K_2)) \to T_1(Y(K_i))$$

induced by injection is an isomorphism. Since $H_1(M) \cong B_1(M) \oplus T_1(M)$ for any manifold $M$, we have an isomorphism

$$\phi_0 = (j_2 \circ j_1^{-1}, k_2 \circ k_1^{-1}) : H_1(Y(K_1)) \cong B_1(Y(K_1)) \oplus T_1(Y(K_1)) \cong B_1(Y(K_2)) \oplus T_1(Y(K_2)) \cong H_1(Y(K_2)).$$

Moreover, if

$$l_i : H_1(Y(K_i)) \to H_1(Y)$$

is the map induced by injection, then $l_1 = \phi_0 \circ l_2$.

For $i = 1, 2$, consider the long exact sequence about the pair $(Y, N(K_i))$:

$$H^1(Y) \to H^1(N(K_i)) \to H^2(Y, N(K_i)) \to H^2(Y) \to H^2(K_i) = 0.$$ 

Since $Y$ is a rational homology sphere,

$$H^1(Y) \cong \text{Hom}(H_1(Y), \mathbb{Z}) = 0.$$ 

By excision theorem and the Poincaré duality, we have

$$H^2(Y, N(K_i)) \cong H^2(Y(K_i), \partial Y(K_i)) \cong H_1(Y(K_i))$$

and $H^2(Y) \cong H_1(Y)$.

Under the Poincaré duality, the image of

$$H^1(N(K_i)) \cong H_1(N(K_i), \partial N(K_i)) \cong \mathbb{Z}$$

in $H_1(Y(K_i))$ is $[m_1]$. Since $l_1 = \phi \circ l_2$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(Y(K_1)) & \xrightarrow{l_1} & H_1(Y) & \longrightarrow & 0 \\
& & & \uparrow \cong & \downarrow \phi_0 & & \downarrow = & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(Y(K_2)) & \xrightarrow{l_2} & H_1(Y) & \longrightarrow & 0
\end{array}$$

Hence $\phi_0([m_1]) = [m_2]$. If $\phi_0([m_1]) = [m_2]$, let $\phi = \phi_0$. If $\phi_0([m_1]) = [m_2]^{-1}$, let $\phi = \phi_0 \circ \epsilon$, where $\epsilon$ maps an element to its inverse. Then $\phi : H_1(Y(K_1)) \to H_1(Y(K_2))$ is an isomorphism and $\phi([m_1]) = ([m_2])$. \qed

Lemma 6.4. Suppose $G$ is an abelian group and $g$ is an element in $G$. The quotient map $G \to G/(g)$ induces a map on the group ring

$$\text{pr} : \mathbb{Z}[G] \to \mathbb{Z}[G/(g)].$$

Then the kernel of $\text{pr}$ is generated by $1 - g$. 

Proof. The functor that takes a group to its group ring is left-adjoint to the functor that takes a commutative ring to its group of units. The quotient $G/(g)$ is the colimit of the diagram $\mathbb{Z} \to G$, where one map is $1 \mapsto g$ and the other is the zero map. Then the proposition follows from the fact that left-adjoints preserve colimits.

Proof of Theorem 6.2. The first part of this theorem is just Lemma 6.3. Write $H = H_1(Y(K_i))$ and $[m] = [m_i]$. By Theorem 4.13 we have

$$\chi(\hat{HFK}(Y, K_i)) = (1 - [m])\tau(K_i) \in \mathbb{Z}[H] \text{ or } \left(\frac{1}{2}\mathbb{Z}\right)[H].$$

A priori, the Turaev torsion $\tau(Y(K_i))$ is not in $\mathbb{Z}[H]$, but the difference $\tau(Y(K_1)) - \tau(Y(K_2))$ is. By Lemma 6.4 we can apply the proof of [Ye21] Lemma 5.5 to the case where $Y$ is a rational homology sphere. Note that we use the fact $b_1(Y(K_i)) = 1$ in that proof. From [Ye21] Lemma 5.5, we have

$$\tau(Y(K_1)) - \tau(Y(K_2)) = (1 - [m])g \in \mathbb{Z}[H] \text{ or } \left(\frac{1}{2}\mathbb{Z}\right)[H] \text{ for some } g \in \mathbb{Z}[H].$$

The ambiguity of $\pm H$ in the statement of [Ye21] Lemma 5.5 is resolved because we consider absolute Alexander gradings on $\hat{HFK}(Y, K_i)$. Then we have

$$\chi(\hat{HFK}(Y, K_1)) - \chi(\hat{HFK}(Y, K_2)) = (1 - [m])(\tau(Y(K_1)) - \tau(Y(K_2))) = (1 - [m])^2 g.$$  

Suppose $H_1(Y; \mathbb{Z}) = \{s_1, \ldots, s_p\}$. Then the element $g$ can be written as the sum

$$g = \sum_{j=1}^{p} g_j,$$

where $g_j$ contains terms that are in the preimage of $s_j \in H_1(Y; \mathbb{Z})$ under the map

$$H \to H/m \cong H_1(Y; \mathbb{Z}).$$

For any $j$, there exists a Laurent polynomial $f_j(x)$ and an element $\hat{s}_j \in H$ such that $g_j = f_j([m])\hat{s}_j$. Since Spin$^c(Y)$ is an affine space on $H_1(Y)$, Thus, the equation (6.1) can be decomposed with respect to Spin$^c(Y)$, where $h_s$ corresponds to some $\hat{s}_j$.

Finally, we deal with instanton knot homology.

Definition 6.5. Suppose $K$ is a knot in a rational homology sphere $Y$ and suppose $H = H_1(Y(K))$. Similar to the way for $\hat{HFK}(Y, K)$, we can fix the $\mathbb{Z}_2$-grading and the $H$-grading from the enhanced Euler characteristic on $KHI(Y, K)$. When gradings are fixed, the space $KHI(Y, K)$ is called also the canonical representative and the corresponding $H$-grading is also called the absolute Alexander grading. For any element $s \in H_1(Y)$, let $[s] \subset H_1(Y(K))$ be the set of preimages of $s$ under the map $H \to H_1(Y)$ and define

$$KHI(Y, K, [s]) := \bigoplus_{h \in [s]} KHI(Y, K, h).$$

Then $\chi(KHI(Y, K, [s]))$ is also a well-defined element in $\mathbb{Z}[H] \text{ or } \left(\frac{1}{2}\mathbb{Z}\right)[H]$. By Theorem 6.2 and the equation (1.1), we have the following corollary.
Corollary 6.6. Suppose $K_1$ and $K_2$ be two knots in a rational homology sphere $Y$ with $[K_1] = [K_2] \in H_1(Y)$. For $i = 1, 2$, suppose $m_i$ is the meridian of $K_i$. Using the isomorphism $\phi$ in Lemma 6.3, we write $H_1(Y(K_i); \mathbb{Z})$ as $H$ and write $[m_i]$ as $[m] \in H$.

Consider the canonical representative of $KHI(Y, K_1)$. Then for any $s \in H_1(Y)$, there exists a Laurent polynomial $f_s(x) \in \mathbb{Z}[x, x^{-1}]$ and an element $h_s \in H$ such that
\[
\chi(KHI(Y, K_1, [s])) - \chi(KHI(Y, K_2, [s])) = ([m] - 1)^2 f_s([m]) h_s,
\]
where both sides are elements in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$.

6.2. Detection results.

In this subsection, we use Theorem 6.2 and Corollary 6.6 to prove detection results in the introduction.

Convention. Throughout this subsection, we suppose $K$ is a knot in a rational homology sphere $Y$ and suppose $H = H_1(Y(K))$. Moreover, we consider canonical representatives of $HF(Y, K)$ and $KHI(Y, K)$ as in Definition 6.1 and Definition 6.5. For simplicity, we write also write $\mathbb{Z}$ as $\mathbb{Z}[H]$.

First, we prove some lemmas.

Lemma 6.7. Suppose $K \subset Y$ is a knot so that $[K] = 0 \in H_1(Y)$. Then there exists a canonical isomorphism
\[
H_1(Y(K)) \cong \mathbb{Z} \oplus H_1(Y),
\]
where the meridian of $K$ represents the generator of $\mathbb{Z}$.

Proof. The isomorphism is induced by pairing with a Seifert surface of $K$. \hfill \Box

Hence we can write elements in $H_1(Y(K))$ as $[m]n \cdot s$ for $s \in H_1(Y)$ and $n \in \mathbb{Z}$.

Lemma 6.8. Suppose $Y$ is an instanton L-space and $U \subset Y$ is the unknot. Then for any $s \in H_1(Y)$, we have
\[
KHI(Y, U, [s]) \cong \mathbb{C} \text{ and } \chi(KHI(Y, U, [s])) = s \in \mathbb{Z}[H].
\]

Proof. The result $KHI(Y, U, [s]) \cong \mathbb{C}$ follows directly from the equation (1.1) and the following isomorphisms:
\[
KHI(Y, U) \cong I^b(Y) \cong \mathbb{C}[H_1(Y)].
\]
The result $\chi(KHI(Y, U, [s])) = 1$ follows from the fact that $g(U) = 0$ and $KHI$ detects the genus of the knot [KM10b, Proposition 7.16]. \hfill \Box

Lemma 6.9. Suppose $Y$ is an instanton L-space, and $K \subset Y$ is a knot so that $[K] = 0 \in H_1(Y)$. Suppose $m$ is the meridian of $K$. Then for any $s \in H_1(Y)$, the element
\[
\chi(KHI(Y, K, [s])) - s \in \mathbb{Z}[H]
\]
has a factor $([m] - 1)^2$.

Proof. Since the unknot is also null-homologous, this lemma follows directly from Corollary 6.6 and Lemma 6.8. \hfill \Box

Then we prove the detect results in the introduction.
Proof of Theorem 1.5. It is clear that
\[ \dim \mathcal{C}(Y,K;[\mathcal{C}(Y,K)]) \neq 0 \]
and we will show that \( K \) must be the unknot. For any \( s \in H_1(Y) \), Lemma 6.9 implies that
\[ \chi(KHI(Y,K,[s])) \neq 0 \]
and hence \( KHI(Y,K,[s]) \neq 0 \). From the assumption, we have
\[ \dim \mathcal{C}(Y,K;[\mathcal{C}(Y,K)]) = \dim \mathcal{I}(Y) = |H_1(Y)|. \]
Thus, we must have
\[ KHI(Y,K,[s]) \cong \mathbb{C} \]
and \( \chi(KHI(Y,K,[s])) = [m] \cdot s \in \mathbb{Z}[H] \),
where \( m \) is the meridian of \( K \) and \( n \) is some integer. Applying Lemma 6.9 again, we know that \( n \) must be 0. Since \( KHI \) detects the genus of the knot \[ \text{[KM10]} \text{ Proposition 7.16} \], we know that \( g(K) = 0 \), which implies \( K \) is the unknot.

Proof of Theorem 1.8. Applying Lemma 6.9 for any \( s \in H_1(Y) \), we have
\[ \chi(KHI(Y,K,[s])) \neq 0. \]
Since \( \chi(KHI(Y,K,[s])) \) and \( \dim \mathcal{C}(Y,K,[s]) \) have the same parity, we conclude that there exists \( s_0 \in H_1(Y) \) so that
\[ KHI(Y,K,[s_0]) \cong \mathbb{C}^3 \]
and for any \( s \neq s_0 \),
\[ KHI(Y,K,[s]) \cong \mathbb{C} \]
Applying Lemma 6.9 again, for any \( s \neq s_0 \), we know that
\[ \chi(KHI(Y,K,[s])) = s \in \mathbb{Z}[H]. \]
For \( s_0 \), we know that
\[ \chi(KHI(Y,K,[s_0])) \]
has a factor \( ([m]-1)^2 \) and
\[ ||\chi(KHI(Y,K,[s_0]))|| \leq 3, \]
where \( m \) is the meridian of \( K \).

Hence there are only two possibilities.

**Case 1.** \( ||\chi(KHI(Y,K,[s_0]))|| = 1 \) and we then conclude that
\[ \chi(KHI(Y,K,[s_0])) = [m] \cdot s_0 \in \mathbb{Z}[H]. \]
Note that \( ||\chi(KHI(Y,K,[s_0]))|| = 1 \) implies that there is a 2-dimensional summand of \( KHI(Y,K) \) whose Euler characteristic is zero. Hence there are further two cases:

**Case 1.1** \( KHI(Y,K,[s_0]) \) is supported in two different Alexander gradings. By assumption, we know that \( KHI(Y,K,[s_0]) \) has a 1-dimensional summand at the Alexander grading \( n \) and has a 2-dimensional summand at the Alexander grading \( n' \) for some \( n' \neq n \). This contradicts the fact that \( KHI(Y,K) \) is symmetric with respect to the Alexander grading.

**Case 1.2.** \( KHI(Y,K,[s_0]) \) is supported solely in the Alexander grading \( n \). By symmetry of \( KHI(Y,K) \), we must have \( n = 0 \). Since \( KHI \) detects the genus of the knot, we know \( K \) is an unknot, which contradicts Theorem 1.5 since \( \dim \mathcal{C}(Y,K) = \dim \mathcal{I}(Y) + 2. \)
Case 2. \( ||\chi(KHI(Y,K,[s_0]))|| = 3 \). By symmetry on \( KHI(Y,K) \), there exists \( n \in \mathbb{N}_+ \) so that 
\[
\chi(KHI(Y,K,[s_0])) = ([m]^n - 1 + [m]^{-n}) \cdot s_0 \in \mathbb{Z}[H].
\]
by [KM10b, Proposition 7.16 and Corollary 7.19], we know that \( K \) is fibred of genus \( n \). By [BS18, Theorem 1.7], we know that \( n = 1 \). Hence \( K \) is a genus-one-fibred knot.

The proofs of the following theorems are similar to those of Theorem 1.5 and Theorem 1.8, but using Theorem 6.2 and [BVV18, Theorem 1.1] instead of Corollary 6.6 and [BS18, Theorem 1.7].

**Theorem 6.10.** Suppose \( K \) is a null-homologous knot in a rational homology sphere \( Y \). If
\[
(6.2) \quad \dim_{\mathbb{F}_2} \widehat{HF}(Y) = \left| H_1(Y;\mathbb{Z}) \right|,
\]
then \( K \) is the unknot if and only if
\[
(6.3) \quad \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K) = \dim_{\mathbb{F}_2} \widehat{HF}(Y).
\]

**Theorem 6.11.** Suppose \( K \) is a null-homologous knot in a rational homology sphere \( Y \). If
\[
(6.4) \quad \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K) = \dim_{\mathbb{F}_2} \widehat{HF}(Y) + 2 = \left| H_1(Y;\mathbb{Z}) \right| + 2,
\]
then \( K \) must be a genus-one-fibred knot.

**Remark 6.12.** Baldwin [Bal08] classified L-spaces that contain null-homologous genus-one-fibred knots. He also computed knot Floer homologies of such knots, which only depend on their Alexander polynomials. The techniques in his classification involve the minus chain complex [OS04c] and the mapping cone formula [OS08, OS11], which is not available in instanton theory yet. For knots in lens spaces, there are more results about genus-one-fibred knots [Mor89, BJK09, Bak14].

**References**


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