Constrained knots in lens spaces

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We study a special family of $(1, 1)$ knots called constrained knots, which includes 2–bridge knots in the 3–sphere S^3 and simple knots in lens spaces. Constrained knots are parametrized by five integers and characterized by the distribution of spin^c structures in the corresponding $(1, 1)$ diagrams. The knot Floer homology $\widehat{HFK}$ of a constrained knot is thin. We obtain a complete classification of constrained knots based on the calculation of $\widehat{HFK}$ and presentations of knot groups. We provide many examples of constrained knots constructed from surgeries on links in S^3, which are related to 2–bridge knots and 1–bridge braids. We also show many examples of constrained knots whose knot complements are orientable hyperbolic 1–cusped manifolds with simple ideal triangulations.

57K10, 57K14, 57K18, 57K31, 57K32

1 Introduction

The main object studied in this paper is a special family of knots in lens spaces called constrained knots. Every knot in a closed 3–manifold can be represented by a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$ (see Ozsváth and Szabó [29, Section 2]),
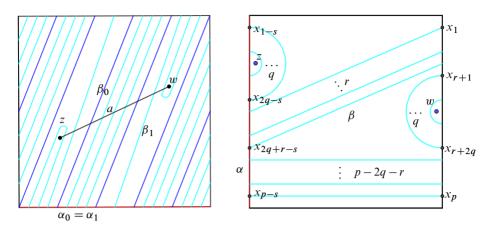
Figure 1: Left: a constrained knot in $L(5, 2)$. Right: a $(1, 1)$ diagram.

where Σ is a closed surface, $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ and $\beta = \{\beta_1, \ldots, \beta_g\}$ are two collections of $g = g(\Sigma)$ simple closed curves on Σ, and z and w are two basepoints on $\Sigma - (\alpha \cup \beta)$. Conversely, any doubly pointed Heegaard diagram defines a knot. Explicitly, the knot is the union of an arc a connecting z to w on $\Sigma - \alpha$, pushed slightly into the α–handlebody, and an arc b connecting w to z on $\Sigma - \beta$, pushed slightly into the β–handlebody.

Let T^2 be the torus obtained by the quotient map $\mathbb{R}^2 \to T^2$ that identifies (x, y) with $(x + m, y + n)$ for $m, n \in \mathbb{Z}$. Suppose p and q are integers satisfying $p > 0$ and $\gcd(p, q) = 1$. Let α_0 and β_0 be two simple closed curves on T^2 obtained from two straight lines in $\mathbb{R}^2$ of slopes 0 and p/q. Then (T^2, α_0, β_0) is called the *standard diagram* of a lens space $L(p, q)$. Let $\alpha_1 = \alpha_0$ and let β_1 be a simple closed curve on T^2 that is disjoint from β_0 and where $[\beta_1] = [\beta_0] \in H_1(T^2; \mathbb{Z})$. Then (T^2, α_1, β_1) is also a Heegaard diagram of $L(p, q)$. Let z and w be two basepoints in $T^2 - \alpha_0 \cup \beta_0 \cup \beta_1$.

The knot defined by the doubly pointed diagram $(T^2, \alpha_1, \beta_1, z, w)$ is called a *constrained knot* and the diagram is called the *standard diagram* of the constrained knot. We will show that constrained knots are parametrized by five integers, which will be denoted by $C(p, q, l, u, v)$. For technical reasons, the knot $C(p, q, l, u, v)$ is in $L(p, q')$, where $qq' \equiv 1 \pmod{p}$. An example is shown in Figure 1, left, where (T^2, α_0, β_0) is the standard diagram of $L(5, 2)$ and $(T^2, \alpha_1, \beta_1, z, w)$ defines $C(5, 3, 2, 3, 1)$.

Roughly speaking, knots defined by doubly pointed Heegaard diagrams with $g(\Sigma) = 1$ are called $(1, 1)$ *knots* and the corresponding diagrams are called $(1, 1)$ *diagrams*; for precise definitions see Section 2.1. These $(1, 1)$ knots are parametrized by four integers (see Goda, Matsuda and Morifuji [14] and Rasmussen [37]), which will be denoted by

$W(p, q, r, s)$; see Figure 1, right. After rotation, standard diagrams of constrained knots are special cases of $(1, 1)$ diagrams. Moreover, the following proposition characterizes constrained knots by the distribution of spin^c structures on the ambient 3–manifold in the corresponding $(1, 1)$ diagrams; for the definition of spin^c structures see Ozsváth and Szabó [29] and Rasmussen [38].

Proposition 1.1 *Let $K = W(p, q, r, s)$ be a $(1, 1)$ knot in $Y = L(a, b)$ with $a > 1$, and suppose $(T^2, \alpha, \beta, z, w)$ is the corresponding $(1, 1)$ diagram of K. Let $\{x_i\}$ be intersection points in $\alpha \cap \beta$, ordered by an orientation of α. Let $\mathfrak{s}_i = \mathfrak{s}_z(x_i) \in \mathrm{Spin}^c(Y)$ be the spin^c structures on Y corresponding to x_i. The knot K is a constrained knot if and only if:*

(i) *For $k = |\mathrm{Spin}^c(Y)| (= a)$, there are integers $p_1, \ldots, p_k$ such that*

$$0 < p_1 < p_2 < \cdots < p_k \le p.$$

(ii) *$\mathfrak{s}_i = \mathfrak{s}_j$ if and only if either $i, j \in (0, p_1] \cup (p_k, p]$, or $i, j \in (p_l, p_{l+1}]$ for some $l \in \{1, \ldots, p - 1\}$.*

A single knot can be represented by $(1, 1)$ knots $W(p_1, q_1, r_1, s_1)$ and $W(p_2, q_2, r_2, s_2)$ with different parameters. For example, both $W(5, 2, 1, 3)$ and $W(5, 2, 1, 0)$ represent the figure-8 knot in S^3. There is no explicit classification of $(1, 1)$ knots by $W(p, q, r, s)$ to the author's knowledge. However, it is possible to classify constrained knots by the parametrization $C(p, q, l, u, v)$. In particular, the case $C(1, 0, 1, u, v)$ consists of 2–bridge knots in S^3 (see Proposition 3.5) and the case $C(p, q, l, 1, 0)$ consists of simple knots in lens spaces (see Proposition 3.7). Schubert [41] and Rasmussen [38] classify 2–bridge knots and simple knots, respectively. The case $C(p, q, 1, u, v)$ consists of connected sums of a core knot in a lens space and a 2–bridge knot (see Theorem 7.14). For other constrained knots, the classification is given by:

Theorem 1.2 *Suppose that $(p_1, q_1, l_1, u_1, v_1)$ and $(p_2, q_2, l_2, u_2, v_2)$ are two different collections of integers satisfying, for $i = 1, 2$,*

$$p_i > 1, \quad q_i \in [1, p_i - 1], \quad l_i \in [2, p_i], \quad u_i > 2v_i > 0, \quad u_i \text{ is odd,}$$
$$\gcd(p_i, q_i) = \gcd(u_i, v_i) = 1.$$

Then constrained knots $C(p_i, q_i, l_i, u_i, v_i)$ represent the same knot if and only if

$$p_1 = p_2 = p, \quad q_1 q_2 \equiv 1 \pmod{p}, \quad l_1, l_2 \in \{2, p\} \quad \text{and} \quad (l_1, u_1, v_1) = (l_2, u_2, v_2).$$

The hat version of knot Floer homology $\widehat{HFK}(Y, K)$ (see Ozsváth and Szabó [29] and Rasmussen [36]) is a powerful invariant for a knot K in a closed 3–manifold Y. It decomposes as the direct sum

$$\widehat{HFK}(Y, K) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HFK}(Y, K, \mathfrak{s}) \tag{1}$$

with respect to spin^c structures on Y. Moreover, the homology $\widehat{HFK}(Y, K, \mathfrak{s})$ inherits two $\mathbb{Z}$–gradings, the Alexander grading and the Maslov grading, from the underlying chain complex $\widehat{CFK}(Y, K, \mathfrak{s})$.

Definition 1.3 A knot $K \subset Y$ is called an $\mathfrak{s}$–*thin* knot if the difference of the Maslov grading and the Alexander grading on $\widehat{HFK}(Y, K, \mathfrak{s})$ is constant for homogeneous elements. It is called a *thin* knot if it is an $\mathfrak{s}'$–thin knot for any $\mathfrak{s}' \in \mathrm{Spin}^c(Y)$.

Thin knots defined as above generalize δ–*thin knots* in S^3 (see Rasmussen [37]) and *Floer homological thin knots*; see Manolescu and Ozsváth [23]. Examples of thin knots include all quasialternating knots [23], in particular all 2–bridge knots.

Suppose K is a thin knot in S^3 and $\mathfrak{s}_0$ is the unique spin^c structure on S^3. Then the minus version of the knot Floer chain complex $CFK^-(S^3, K) = CFK^-(S^3, K, \mathfrak{s}_0)$ is determined by the Alexander polynomial $\Delta_K(t)$ and the signature $\sigma(K)$ up to chain homotopy; see Petkova [34]. For a compact 3–manifold M with torus boundary, there exists a set of immersed curves $\widehat{HF}(M)$ on $\partial M - \mathrm{pt}$, called the *curve invariant* (see Hanselman, Rasmussen and Watson [16; 17]) of M, which encodes the information of Heegaard Floer theory in a diagrammatic way. Based on [17, Section 4; 34, Section 3], it is easy to draw $\widehat{HF}(E(K))$ of the knot complement $E(K) = S^3 - \mathrm{int}\, N(K)$ for a thin knot $K \subset S^3$. Roughly speaking, it consists of figure-8 curves and a distinguished curve.

For a $(1, 1)$ knot $K \subset Y$ there is a combinatorial method to calculate the chain complex $CFK^-(Y, K)$; see Goda, Matsuda and Morifuji [14]. It applies well to 2–bridge knots and also constrained knots. From the standard diagram of a constrained knot $K \subset Y$, if we focus on intersection points corresponding to the same spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, we can obtain an explicit relation between $CFK^-(Y, K, \mathfrak{s})$ and $CFK^-(S^3, K', \mathfrak{s}_0)$, where K' is some 2–bridge knot. In particular, for $K = C(p, q, l, u, v) \subset Y$ and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, the group $\widehat{HFK}(Y, K, \mathfrak{s})$ is determined by Alexander polynomials of 2–bridge knots $K_1 = \mathfrak{b}(u, v)$ and $K_2 = \mathfrak{b}(u - 2v, v)$. Hence:

Proposition 1.4 *Constrained knots are thin.*

Results about thin complexes in [34, Section 3] apply directly to $CFK^-(Y, K, \mathfrak{s})$ for a constrained knot $K \subset Y$. Then we can draw the part of the curve invariant corresponding to each spin^c structure following the approach in [17, Section 4]. Similar to the case of a 2–bridge knot, the curve invariant part consists of figure-8 curves and a distinguished curve; see Figure 7.

To connect the distinguished curves of different parts, we should study the following grading, which relates elements in $\widehat{HFK}(Y, K, \mathfrak{s})$ for different spin^c structures. The total homology $\widehat{HFK}(Y, K)$ inherits a relative $H_1(E(K); \mathbb{Z})$–grading (see Rasmussen [38, Section 3.3]):

$$\text{(2)} \qquad \widehat{HFK}(Y, K) = \bigoplus_{h \in H_1(E(K); \mathbb{Z})} \widehat{HFK}(Y, K, h).$$

This grading generalizes the Alexander grading on $\widehat{HFK}(S^3, K, \mathfrak{s}_0)$ and corresponds to spin^c structures on $E(K)$ with some boundary conditions; see Juhász [20, Section 4]. Under the map $H_1(E(K); \mathbb{Z}) \to H_1(Y; \mathbb{Z})$, this grading reduces to the grading in (1).

Similar to the Alexander grading on $\widehat{HFK}(S^3, K, \mathfrak{s}_0)$, summands of $\widehat{HFK}(Y, K)$ in the opposite $H_1(E(K); \mathbb{Z})$–gradings are isomorphic (see Section 3 of Ozsváth and Szabó [29]), up to a global grading shift in $H_1(E(K); \mathbb{Z})$. This symmetry is called the *global symmetry*. If it is not mentioned, this $H_1(E(K); \mathbb{Z})$–grading is also called the *Alexander grading*, and denoted by $\text{gr}(x) \in H_1(E(K); \mathbb{Z})$ for a homogeneous element $x \in \widehat{HFK}(Y, K)$. To fix the ambiguity of the global grading shift, a specific grading shift will be used so that under the global symmetry, the absolute value of the Alexander grading is left invariant. The Alexander grading in this specific grading shift is called the *absolute Alexander grading*. To be clear, when considering the Alexander grading on $\widehat{HFK}(Y, K, \mathfrak{s})$ mentioned before, the spin^c structure $\mathfrak{s}$ will be specified.

Following [38, Section 3.3], for a constrained knot, the Alexander grading can be calculated from the standard diagram. The Alexander grading on $\widehat{HFK}(Y, K)$ indicates an explicit way to connect different parts of the curve invariant. Then it is not hard to draw the whole curve invariant of a constrained knot. As an application, much information about the Heegaard Floer theory of a constrained knot can be obtained from the curve invariant of the knot complement.

For a constrained knot $K \subset Y$ and the corresponding 2–bridge knots K_1 and K_2 mentioned before, the symmetry on $\widehat{HFK}(S^3, K_i, \mathfrak{s}_0)$ for $i = 1, 2$ induces a symmetry

on $\widehat{HFK}(Y, K, \mathfrak{s})$, which is called the *local symmetry*. For $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, the average $A(K, \mathfrak{s})$ of any two homogeneous elements $x, y \in \widehat{HFK}(Y, K, \mathfrak{s})$ that are symmetric under the local symmetry is called the *middle grading* of $\mathfrak{s}$:

$$A(K, \mathfrak{s}) = \tfrac{1}{2}(\mathrm{gr}(x) + \mathrm{gr}(y)) \in H_1(E(K); \mathbb{Z}).$$

Theorem 1.5 *For $i = 1, 2$ let $K_i = C(p_i, q_i, l_i, u_i, v_i)$ be constrained knots in the same lens space Y with $[K_1] = [K_2] \in H_1(Y; \mathbb{Z})$. Consider the absolute Alexander grading on $\widehat{HFK}(Y, K_i)$. Then there are isomorphisms*

$$H_1(E(K_1); \mathbb{Z}) \cong H_1(E(K_2); \mathbb{Z}) \cong H_1,$$

so that $A(K_1, \mathfrak{s}) = A(K_2, \mathfrak{s}) \in H_1$ for any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

Theorem 1.6 *Suppose K is a knot in $Y = L(p, q)$. Let K' be a simple knot in the same manifold Y with $[K'] = [K] \in H_1(Y; \mathbb{Z})$. Consider the absolute Alexander gradings on $\widehat{HFK}(Y, K)$ and $\widehat{HFK}(Y, K')$. We know $\widehat{HFK}(Y, K') \cong \mathbb{Z}^p$. If $\widehat{HFK}(Y, K) \cong \mathbb{Z}^p$, then there are isomorphisms $H_1(E(K); \mathbb{Z}) \cong H_1(E(K'); \mathbb{Z}) \cong H_1$, so that there is a one-to-one correspondence between generators of $\widehat{HFK}(Y, K)$ and $\widehat{HFK}(Y, K')$ with the same absolute Alexander grading in H_1.*

Theorem 1.6 provides a clue for the following conjecture, which is related to Berge's conjecture [2] claiming that any knot in S^3 admitting lens space surgeries falls into Berge's list.

Conjecture 1.7 (Baker, Grigsby and Hedden [1] and Hedden [19]) *Suppose K is a knot in $Y = L(p, q)$. If $\widehat{HFK}(Y, K) \cong \mathbb{Z}^p$, then K is a simple knot, ie a $(1, 1)$ knot $W(p', q', r', s')$ with $q' = 0$.*

Though constrained knots are defined by doubly pointed Heegaard diagrams, there are many other ways to construct constrained knots, at least for some special families of parameters. In the following, we introduce two approaches based on Dehn surgeries on links in S^3.

The first approach is inspired by the relation between knot Floer homologies of constrained knots and 2–bridge knots. A *magic link* is a 3–component link as shown in Figure 2, left, where K_0 is a 2–bridge knot, and K_1 and K_2 are unknots. Dehn surgeries on K_1 and K_2 induce a lens space, in which K_0 becomes a knot K_0'.
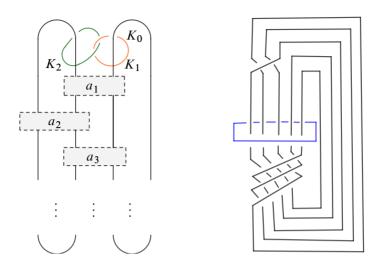
Figure 2: Left: magic link. Right: 1–bridge braid.

Theorem 1.8 *Suppose that integers p and q satisfy $p > q > 0$ and $\gcd(p, q) = 1$. Suppose integers n_1, n_2 and l satisfy*

$$n_1 \in \left[0, \frac{p}{q}\right), \quad n_2 \in \left[0, \frac{p}{p-q}\right),$$
$$l \in \{n_1 q + 1, p - n_1 q + 1, n_2(p-q) + 1, p - n_2(p-q) + 1\}.$$

Let $L = K_0 \cup K_1 \cup K_2$ be a magic link with $K_0 = \mathfrak{b}(u, v)$. Then the knot $C(p, q, l, u, v)$ is equivalent to the knot K_0' obtained by performing some Dehn surgeries on K_1 and K_2.

The second approach arises from 1–bridge braids. Suppose the solid torus $H = S^1 \times D^2$ is embedded in $\mathbb{R}^3 \subset S^3$ in a standard way, and suppose K_1 is the core of $S^3 - H$. Let $K_0 \subset H$ be a 1–bridge braid; see Gabai [12; 13]. Then $L = K_0 \cup K_1$ is a 2–component link in S^3; an example is given in Figure 2, right. Dehn filling along a simple closed curve on ∂H is equivalent to Dehn surgery on K_1. The resulting manifold is a lens space and K_0 becomes a knot K_0' in the lens space. A knot K_0' constructed from this approach is called a 1–*bridge braid knot*.

Theorem 1.9 *The knots $C(p, q, l, u, \pm 1)$ are equivalent to 1–bridge braid knots, where $C(p, q, l, u, -1)$ means $C(p, q, l, u, u - 1)$.*

Other than Dehn surgeries, constrained knots can also be constructed by Dehn filling the boundary of (orientable hyperbolic) 1–cusped manifolds. Many 1–cusped manifolds

are knot complements of constrained knots. *SnapPy* by Culler, Dunfield and Weeks [7] provides a list of 59 068 1–cusped manifolds admitting ideal triangulations with at most nine tetrahedra. Using the code in Ye [45], we show 21 922 of them are complements of constrained knots. Table 1 shows examples of 1–cusped manifolds that are complements of constrained knots. The names of manifolds in the table are from *SnapPy*. The slopes in the table are considered in the basis from *SnapPy* and the integers indicate the parametrization of the constrained knot that is equivalent to the core of the filling solid torus. For example, Dehn filling along the curve of slope $1/0$ on the boundary of $m003$ gives $C(10, 3, 3, 1, 0)$. If different parametrizations correspond to the same knot (see Theorem 1.2), we only show one collection of parameters. The complete list can be found in [45].

Proposition 1.10 *Curve invariants* $\widehat{HF}(M)$ *of knot complements* M *of all 1–cusped manifolds that have ideal triangulations with at most 5 tetrahedra can be drawn explicitly, except the manifolds in* Table 2.

Proof There are 286 orientable 1–cusped manifolds that have ideal triangulations with at most five ideal tetrahedra. Of these, 232 manifolds are complements of constrained knots, whose curve invariants can be calculated by the method in Section 4. Other than examples from constrained knots, 37 manifolds are Floer simple (by the list in Dunfield [10]), whose curve invariants can be calculated by the approach in Hanselman, Rasmussen and Watson [17, Section 1]. Other manifolds are listed in Table 2 ($(1, 1)$ parameters are from Dunfield's code [45]). The chain complex $CFK^-(Y, K)$ of a $(1, 1)$ knot can be calculated by the method in Goda, Matsuda and Morifuji [14]. Then the curve invariant can be calculated by [17, Section 4]. Note that chain complexes of 8_{20}, 9_{42} and 9_{46} in the table were calculated in Ozsváth and Szabó [32]. □

It is known that a 2–bridge knot $\mathfrak{b}(u, v)$ is a torus knot if $v = 1$ or $v = u - 1$. The latter case is written as $v = -1$. If $v \neq \pm 1$, the 2–bridge knot $\mathfrak{b}(u, v)$ is hyperbolic, that is, the interior of the knot complement admits a hyperbolic metric of finite volume. We may generalize the results about 2–bridge knots to constrained knots. Note that the knot complement of a torus knot is a Seifert fibered space.

Theorem 1.11 *If* $C(p, q, l, u, v)$ *has Seifert fibered complement, then* $v = \pm 1$.

Since $C(p, q, 1, u, v)$ is a connected sum of two knots, there is an essential torus in the knot complement, and hence $C(p, q, 1, u, v)$ is not hyperbolic. Using the code in [45]

name	slope $+ (p, q, l, u, v)$
$m003$	$(1, 0) + (10, 3, 3, 1, 0), (-1, 1) + (5, 4, 5, 3, 1), (0, 1) + (5, 4, 5, 3, 1)$
$m004$	$(1, 0) + (1, 0, 1, 5, 2)$
$m006$	$(0, 1) + (15, 4, 2, 1, 0), (1, 0) + (5, 3, 4, 3, 1)$
$m007$	$(1, 0) + (3, 1, 2, 3, 1)$
$m009$	$(1, 0) + (2, 1, 2, 5, 2)$
$m010$	$(1, 0) + (6, 5, 6, 3, 1)$
$m011$	$(1, 0) + (13, 3, 3, 1, 0), (0, 1) + (9, 4, 9, 3, 1)$
$m015$	$(1, 0) + (1, 0, 1, 7, 2)$
$m016$	$(0, 1) + (18, 5, 3, 1, 0), (-1, 1) + (19, 7, 2, 1, 0)$
$m017$	$(0, 1) + (14, 3, 5, 1, 0), (-1, 1) + (21, 8, 21, 1, 0), (1, 0) + (7, 5, 6, 3, 1)$
$m019$	$(0, 1) + (17, 5, 4, 1, 0), (1, 1) + (11, 7, 11, 3, 1), (1, 0) + (6, 5, 5, 3, 1)$
$m022$	$(1, 0) + (7, 6, 7, 3, 1)$
$m023$	$(1, 0) + (3, 1, 3, 5, 2)$
$m026$	$(0, 1) + (19, 4, 2, 1, 0), (1, 0) + (8, 3, 7, 3, 1)$
$m027$	$(1, 0) + (16, 3, 3, 1, 0), (0, 1) + (13, 4, 13, 3, 1)$
$m029$	$(1, 0) + (5, 2, 3, 3, 1)$
$m030$	$(1, 0) + (7, 4, 5, 3, 1)$
$m032$	$(1, 0) + (1, 0, 1, 9, 2)$
$m033$	$(0, 1) + (18, 5, 5, 1, 0), (1, 0) + (9, 7, 8, 3, 1)$
$m034$	$(1, 0) + (4, 1, 3, 3, 1)$
$m035$	$(1, 0) + (4, 1, 2, 3, 1)$
$m036$	$(-1, 1) + (21, 8, 2, 1, 0), (1, 0) + (3, 2, 3, 5, 1)$
$m037$	$(1, 1) + (24, 7, 2, 1, 0), (1, 0) + (8, 5, 6, 3, 1)$
$m038$	$(1, 0) + (3, 2, 3, 5, 2)$
$m039$	$(1, 0) + (4, 1, 4, 5, 2)$
$m040$	$(1, 0) + (8, 7, 8, 3, 1)$
$m043$	$(0, 1) + (25, 7, 24, 1, 0), (-1, 1) + (25, 9, 2, 1, 0)$
$m044$	$(0, 1) + (24, 7, 23, 1, 0), (-1, 1) + (17, 10, 17, 3, 1), (1, 0) + (7, 6, 5, 3, 1)$
$m045$	$(1, 0) + (2, 1, 2, 7, 2)$
$m046$	$(-1, 1) + (30, 11, 30, 1, 0), (1, 0) + (10, 7, 8, 3, 1)$
$m047$	$(0, 1) + (23, 4, 2, 1, 0), (1, 0) + (11, 3, 10, 3, 1)$
$m049$	$(1, 0) + (19, 3, 3, 1, 0), (0, 1) + (17, 13, 17, 3, 1)$
$m052$	$(0, 1) + (26, 7, 3, 1, 0), (1, 0) + (7, 5, 4, 3, 1)$
$m053$	$(1, 0) + (1, 0, 1, 11, 2)$
$m054$	$(0, 1) + (22, 5, 7, 1, 0), (1, 0) + (11, 8, 9, 3, 1)$
$m055$	$(1, 0) + (23, 7, 5, 1, 0), (0, 1) + (14, 11, 13, 3, 1)$

Table 1: 1–cusped manifolds and constrained knots.

name	comments	name	comments
$m136$	no lens space filling	$m305$	no lens space filling
$m137$	$W(8,2,3,1) \subset S^1 \times S^2$	m306	no lens space filling
$m199$	$9_{42} = W(9,2,2,3) \subset S^3$	$m345$	$W(10,3,1,5) \subset L(2,1)$
$m201$	$10_{132} = W(11,2,1,3) \subset S^3$	$m370$	$(1,0)$ filling gives $L(8,3)$
$m206$	$(1,0)$ filling gives $L(5,2)$	$m372$	$9_{46} = \text{Pretzel}(-3,3,3) \subset S^3$
$m222$	$8_{20} = W(9,3,0,2) \subset S^3$	$m389$	$10_{139} = W(11,3,1,4) \subset S^3$
$m224$	$11_{190} = W(13,2,1,8) \subset S^3$	$m390$	$(1,0)$ filling gives $L(7,2)$
$m235$	no lens space filling	$m410$	no lens space filling
$m304$	$W(12,3,0,5) \subset L(2,1)$		

Table 2: Exceptions of 1–cusped manifolds.

and the *verify_hyperbolicity*() function in *SnapPy*, we verified that $C(p,q,l,u,v)$ is hyperbolic for $p \leq 10$, $l > 1$, $u < 20$ and $v \neq \pm 1$.

Conjecture 1.12 $C(p,q,l,u,v)$ with $l > 1$ and $v \neq \pm 1$ is hyperbolic.

Organization The remainder of this paper is organized as follows. In Section 2, we collect some conventions and definitions in 3–dimensional topology, and facts about $(1,1)$ knots, simple knots and 2–bridge knots. In Section 3, we describe the parametrization of constrained knots and prove Proposition 1.1. Many propositions about constrained knots are also given in Section 3. In Section 4, an algorithm for the knot Floer homology of a constrained knot is obtained, which induces Proposition 1.4 and the necessary part of Theorem 1.2. In Section 5, we study knots in the same homology class and prove Theorems 1.5 and 1.6 by Turaev torsions of 3–manifolds. In Section 6, we finish the proof of Theorem 1.2 by constructing isomorphisms between fundamental groups of knot complements and applying the fact that knots are determined by their fundamental groups. The last three sections discuss magic links, 1–bridge braid knots and SnapPy manifolds, respectively.

would also like to thank his parents and relatives for their support. Calculations are based on *Mathematica*, *SageMath* [40] and *SnapPy* [7].

2 Preliminaries

We begin with basic conventions. For $r \in \mathbb{R}$, let $\lceil r \rceil$ and $\lfloor r \rfloor$ denote the minimum integer and the maximum integer satisfying $\lceil r \rceil \geq r$ and $\lfloor r \rfloor \leq r$, respectively. For a group H, let Tors H denote the set of torsion elements in H.

If it is not mentioned, all manifolds are smooth, connected and oriented, and orientations of knots are omitted. The fundamental group of a manifold M is denoted by $\pi_1(M)$, where the basepoint is omitted. For a submanifold A in a manifold Y, let $N(A)$ denote the regular neighborhood of A in Y and let int $N(A)$ denote its interior. Suppose Y is a closed 3–manifold and K is a knot in Y. Let $E(K) = Y - \text{int } N(K)$ denote the knot complement of K.

For a simple closed curve α on a surface Σ, let $[\alpha]$ denote its homology class in $H_1(\Sigma; \mathbb{Z})$. If it is clear, we do not distinguish α and $[\alpha]$. The algebraic intersection number of two curves α and β on a surface Σ is denoted by $[\alpha] \cdot [\beta]$ or $\alpha \cdot \beta$, while the number of intersection points of α and β is denoted by $|\alpha \cap \beta|$.

A basis (m, l) of $H_1(T^2; \mathbb{Z})$ always satisfies $m \cdot l = -1$. Suppose K is a knot in a closed 3–manifold Y. A basis of $\partial E(K)$ means a basis of $H_1(\partial E(K); \mathbb{Z}) \cong H_1(T^2; \mathbb{Z})$. In practice, there are two standard choices of the basis of $\partial E(K)$:

(i) Let m and l be simple closed curves on $\partial E(K)$ such that Dehn filling along m gives Y, $m \cdot l = -1$, and the orientation of m is determined from the orientation of K by the "right-hand rule". The curves m and l are called the *meridian* and the *longitude* of the knot K, respectively. The basis (m, l) is called the *regular basis* of $\partial E(K)$.

(ii) Let m^* and l^* be simple closed curves on $\partial E(K)$ such that l^* represents the generator of $\text{Ker}(H_1(E(K); \mathbb{Q}) \to H_1(Y; \mathbb{Q}))$ and $m^* \cdot l^* = -1$. They are called the *homological meridian* and the *homological longitude* of the knot K, respectively. The basis (m^*, l^*) is called the *homological basis* of $\partial E(K)$.

The choices of l and m^* are not unique. The longitude l is isotopic to K, while m^* does not have any geometric meaning. Sometimes (eg for knots in S^3) these two choices of the basis are equivalent. If it is not mentioned, we choose the regular basis (m, l) as

the basis of $\partial E(K)$. The slope p/q of a Dehn surgery indicates that the meridian of the filling solid torus is glued to the curve corresponding to $pm + ql$.

Suppose M is an oriented manifold. Let $-M$ denote the same manifold with the reverse orientation, called the *mirror manifold* of M. Suppose K is an (oriented) knot in a 3–manifold M. Then it is specified by the knot complement $E(K)$ and the (oriented) meridian m of the knot. The *mirror image* of K is the knot in $-M$ specified by $(-M, -m)$.

When mentioning that $Y = L(p, q)$ is a lens space, we always suppose that p and q are integers satisfying $\gcd(p, q) = 1$ and $(p, q) \neq (0, 1)$. In particular, the manifold $S^1 \times S^2$ is not considered as a lens space. The lens space is oriented as follows. Let (T^2, α_0, β_0) be the standard diagram of a lens space. Then the orientation on the α_0–handlebody is induced from the standard embedding of a solid torus in $\mathbb{R}^3$. With this convention, the lens space $L(p, q)$ is obtained from the p/q Dehn surgery on the unknot in S^3.

We recall some definitions about knots in closed 3–manifolds. Suppose K is a knot in a lens space Y.

The knot K is called a *trivial knot* or an *unknot* if it bounds a disk embedded in Y. It is called a *core knot* if $E(K)$ is homeomorphic to a solid torus. It is called a *split knot* if Y contains a sphere which decomposes Y into a punctured lens space and a ball containing K in its interior. It is called a *composite knot* if Y contains a 2–sphere S which intersects K transversely in two points and $S \cap E(K)$ is ∂–incompressible in $E(K)$. It is called a *prime knot* if it is not a composite knot.

The torus $T^2 \subset Y$ in the standard diagram (T^2, α_0, β_0) is called the *Heegaard torus* of Y. The knot K is called a (p, q) *torus knot in* Y if K can be isotoped to lie on the Heegaard torus as an essential curve with slope p/q in the standard diagram of Y. The unknot is considered as a torus knot. Complements of torus knots in lens spaces are Seifert fibered spaces.

The knot K is called a *satellite knot* if $E(K)$ has an essential torus. For $q > 1$, the space $C_{p,q}$ is obtained by removing a regular fiber from a solid torus with a (p, q) fibering, which is called a *cable space of type* (p, q). The knot K is called a (p, q) *cable knot* on K_0 if K_0 is knot in Y such that $E(K) = E(K_0) \cup C_{p,q}$. In this case, the knot K lies as an essential curve on $\partial N(K_0)$, and K is neither a longitude nor a meridian of K_0. It is well-known that composite knots are satellite knots. A cable knot on K_0 with $E(K_0)$ having an incompressible boundary is also a satellite knot.

2.1 (1, 1) knots

In this subsection, we review some facts about $(1, 1)$ knots. Proofs are omitted.

A knot K in a closed 3–manifold Y has *tunnel number one* if there is a properly embedded arc γ in $E(K)$ such that $E(K) - N(\gamma)$ is a genus two handlebody. Equivalently, the knot complement $E(K)$ admits a genus two Heegaard splitting. The arc γ is called an *unknotting tunnel* for K. A properly embedded arc γ in a handlebody H is called a *trivial arc* if there is an embedded disk $D \subset H$ such that $\partial D = \gamma \cup (D \cap \partial H)$. The disk D is called the *canceling disk* of γ. A knot K in a 3–manifold Y admits a $(1, 1)$ *decomposition* if there is a genus one Heegaard splitting $Y = H_1 \cup_{T^2} H_2$ such that $K \cap H_i$ is a properly embedded trivial arc k_i in H_i for $i = 1, 2$. In this case, Y is either a lens space (including S^3), or $S^1 \times S^2$. A knot K that admits a $(1, 1)$ decomposition is called a $(1, 1)$ *knot*. We do not consider $(1, 1)$ knots in $S^1 \times S^2$. Note that any $(1, 1)$ knot has tunnel number one.

Proposition 2.1 [44, Proposition 3.2] *If a nontrivial knot in a lens space has tunnel number one, then the complement is irreducible. Consequently, the complement is a Haken manifold.*

Doubly pointed Heegaard diagrams parametrize their corresponding $(1, 1)$ knots. The orientation of the knot is unimportant in this paper so we may swap the two basepoints.

Proposition 2.2 [14; 37] *For $p, q, r, s \in \mathbb{N}$ satisfying $2q + r \leq p$ and $s < p$, a $(1, 1)$ decomposition of a knot determines and is determined by a doubly pointed Heegaard diagram. After isotopy, such a diagram looks like $(T^2, \alpha, \beta, z, w)$ in Figure 1, right, where p is the total number of intersection points, q is the number of strands around each basepoint, r is the number of strands in the middle band, and the ith point on the right-hand side is identified with the $(i + s)^{th}$ point on the left-hand side.*

Let $W(p, q, r, s) = W(p, q, r, s)_+$ denote the $(1, 1)$ knot defined by Figure 1, right, and let $W(p, q, r, s)_-$ denote the knot defined by the diagram that is vertically symmetric to Figure 1, right. These doubly pointed Heegaard diagrams are called $(1, 1)$ *diagrams*. In the diagrams, strands around basepoints are called *rainbows* and strands in the bands are called *stripes*. The roles of the curves α and β here are different from in [37]. For the same parameters, the knot $W(p, q, r, s)$ is the mirror image of $K(p, q, r, s)$ in [37].

Proposition 2.3 *There are relations among $(1, 1)$ knots:*

(i) $W(p, q, r, s)_+$ *is the mirror image of* $W(p, q, r, p - s)_-$.

(ii) $W(p, q, r, s)_+$ is equivalent to $W(p, q, p - 2q - r, s - 2q)_-$.

Thus, we know that $W(p, q, r, s)_+$ is the mirror image of $W(p, q, p-2q-r, p-s+2q)_+$.

Proof   The first relation is from the vertical symmetry. The second relation is from redrawing the diagram so that the lower band becomes the middle band and the middle band becomes the lower band.   □

Definition 2.4   For a closed 3–manifold Y, consider the hat version of Heegaard Floer homology $\widehat{HF}(Y)$ defined in [31]. A closed 3–manifold Y is called an *L-space* if $\widehat{HF}(Y, \mathfrak{s}) \cong \mathbb{Z}$ for any $\mathfrak{s} \in \text{Spin}^c(Y)$. A knot K in an L-space Y is called an *L-space knot* if a nontrivial Dehn surgery on K gives an L-space.

Theorem 2.5   [15, Theorem 1.2]   *A $(1, 1)$ knot is an L-space knot if and only if, in the corresponding $(1, 1)$ diagram with any orientation of β, all of rainbows around a fixed basepoint are oriented in the same way.*

Definition 2.6   [38, Section 2.1]   Let (T^2, α_0, β_0) be the standard Heegaard diagram of $L(p, q)$ and let P_i for $i \in \mathbb{Z}/p\mathbb{Z}$ be components of $T^2 - \alpha_0 \cup \beta_0$, ordered from left to right. Let $z \in P_1$ and $w \in P_{k+1}$ be two points. The knot defined by $(T^2, \alpha_0, \beta_0, z, w)$ is called a *simple knot*, and is denoted by $S(p, q, k)$ (or by $K(p, q, k)$ in [38]). The orientation of the knot is induced by the orientation of the arc connecting z to w.

Proposition 2.7   [38, Lemma 2.5]   *There are relations among simple knots $S(p, q, k)$:*

(i)   *$S(p, q, -k)$ is the orientation-reverse of $S(p, q, k)$.*

(ii)   *$S(p, -q, -k)$ is the mirror image of $S(p, q, k)$.*

(iii)   *$S(p, q, k) \cong S(p, q', kq')$, where $qq' \equiv 1 \pmod{p}$.*

Note that a simple knot is homotopic to an immersed curve on T^2. The homology class $[S(p, q, k)]$ in $H_1(L(p, q); \mathbb{Z})$ is $k[b]$, where b is the core curve of the β_0–handlebody. The simple knots $S(p, q, k_1)$ and $S(p, q, k_2)$ represent the same homology class if and only if $k_1 \equiv k_2 \pmod{p}$. Thus, there is no relation other than those in Proposition 2.7.

2.2   2–Bridge knots

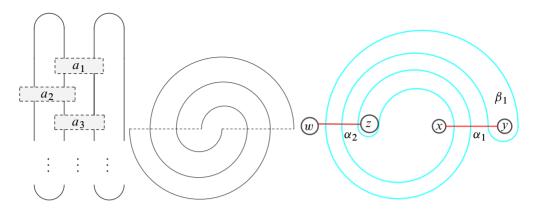In this subsection, we review some facts about 2–bridge links from [6; 25; 35].

Figure 3: Left: 2–bridge. Center: $\mathfrak{b}(3,1)$. Right: diagram of $E(\mathfrak{b}(3,1))$.

Definition 2.8 Suppose h is the height function given by the z–coordinate in $\mathbb{R}^3 \subset S^3$. A knot or a link in S^3 is called a 2–*bridge knot* or a 2–*bridge link* if it can be isotoped in a presentation so that h has two maxima and two minima on it. Such a presentation is called the *standard presentation* of the knot.

A 2–bridge link has two components. Each component is equivalent to the unknot. Suppose integers a and b satisfy $\gcd(a,b) = 1$ and $a > 1$. For every oriented lens space $L(a,b)$, there is a unique 2–bridge knot or link whose branched double cover space is diffeomorphic to $L(a,b)$. Let $\mathfrak{b}(a,b)$ denote the knot or link related to $L(a,b)$. It is a knot if a is odd, and a link if a is even. Thus, the classification of 2–bridge knots or links depends on the classification of lens spaces [5]. For $i = 1, 2$, two 2–bridge knots or links $\mathfrak{b}(a_i, b_i)$ are equivalent if and only if $a_1 = a_2 = a$ and $b_1 \equiv b_2^{\pm 1} \pmod{a}$.

Suppose a/b is represented as the continued fraction

$$[0; a_1, -a_2, \ldots, (-1)^{m+1} a_m] = 0 + \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}}.$$

Moreover, suppose m is odd. The standard presentation of a 2–bridge knot or link $\mathfrak{b}(a,b)$ looks like Figure 3, left, where the $|a_i|$ for $i \in [1, m]$ represent numbers of half-twists in the boxes, and signs of the a_i represent signs of half-twists. Different choices of continued fractions give the same knot or link. For any 2–bridge knot or link, the numbers $(-1)^{i+1} a_i$ can be all positive, which implies any 2–bridge knot or link is alternating.

The knot or link $\mathfrak{b}(a,b)$ admits another canonical presentation known as the *Schubert normal form*. It induces a Heegaard diagram of $E(\mathfrak{b}(a,b))$ and a doubly pointed

Heegaard diagram of $\mathfrak{b}(a, b)$. Figure 3, center, gives an example of the Schubert normal form of $\mathfrak{b}(3, 1)$ and Figure 3, right, is the corresponding Heegaard diagram of the knot complement. The corresponding doubly pointed Heegaard diagram is obtained by replacing α_2 by two basepoints, z and w. Two horizontal strands in the Schubert normal form are arcs near two maxima in the standard presentation. Thus, two 1–handles attached to points w and z, and x and y in Figure 3, right, are neighborhoods of these arcs.

Proposition 2.9 [35] *Suppose $K = \mathfrak{b}(a, b)$ with b odd and $|b| < a$. The symmetrized Alexander polynomial $\Delta_K(t)$ and the signature $\sigma(K)$ satisfy*

$$\Delta_K(t) = t^{-\frac{1}{2}\sigma(K)} \sum_{i=0}^{a-1} (-1)^i t^{\sum_{j=0}^{i}(-1)^{\lfloor ib/a \rfloor}} \quad and \quad \sigma(K) = \sum_{i=1}^{a-1} (-1)^{\lfloor ib/a \rfloor}.$$

Proposition 2.10 [8; 18] *Let K be a $(1, 1)$ knot in a lens space. Then K is a split knot if and only if K is the unknot. The knot K is a composite knot if and only if it is a connected sum of a 2–bridge knot and a core knot of a lens space.*
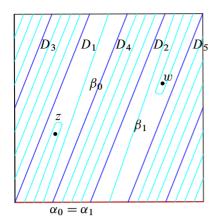
3   Parametrization and characterization

For a constrained knot K, there is a standard diagram $(T^2, \alpha_1, \beta_1, z, w)$ of K, defined in the introduction. Based on standard diagrams, we describe the parametrization of constrained knots. For integers p, q and q' satisfying

$$\gcd(p, q) = \gcd(p, q') = 1 \quad and \quad qq' \equiv 1 \pmod{p}$$

we know that $L(p, q)$ is diffeomorphic to $L(p, q')$ [5]. Suppose (T^2, α_0, β_0) is the standard diagram of $L(p, q')$, ie the curve β_0 is obtained from a straight line of slope p/q' in $\mathbb{R}^2$, and suppose that the diagram $(T^2, \alpha_1, \beta_1, z, w)$ is induced by (T^2, α_0, β_0) as in the introduction. The curves α_0 and β_0 divide T^2 into p regions, which are parallelograms in Figure 1, left; see also Figure 4, left. A new diagram C is obtained by gluing top edges and bottom edges of parallelograms. We can shape C into a square. An example is shown in Figure 4, where $p = 5$, $q = 3$ and $q' = 2$.

For $i \in \mathbb{Z}/p\mathbb{Z}$, let D_i denote rectangles in C, ordered from the bottom edge to the top edge. Since $qq' \equiv 1 \pmod{1}$ and we start with the standard diagram of $L(p, q')$, we know that the right edge of D_j is glued to the left edge of D_{j+q}. The bottom edge e_b of D_1 is glued to the top edge e_t of D_p. By definition of a constrained knot, the curve
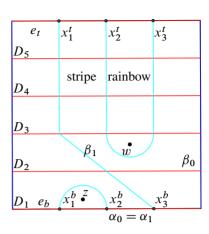
Figure 4: Heegaard diagrams of $C(5, 3, 2, 3, 1)$.

α_1 is the same as α_0 and the curve β_1 is disjoint from β_0. Thus, in this new diagram C, the curve α_1 is the union of p horizontal lines and β_1 is the union of strands which are disjoint from vertical edges of D_i for $i \in \mathbb{Z}/p\mathbb{Z}$.

Similar to the definitions for $(1, 1)$ knots, strands in the standard diagram of a constrained knot are called *rainbows* and *stripes*. Boundary points of a rainbow and a stripe are called *rainbow points* and *stripe points*, respectively. A rainbow must bound a basepoint, otherwise it can be removed by isotopy. Numbers of rainbows on e_b and e_t are the same since the numbers of rainbow points are the same. Without loss of generality, suppose z is in all rainbows on e_b and w is in all rainbows on e_t. Let x_i^b and x_i^t for $i \in [1, u]$ be boundary points on the bottom edge and the top edge, respectively, ordered from left to right in Figure 4, right.

Lemma 3.1 *The number u of boundary points on e_b or e_t is odd. When $u = 1$, there is no rainbow and only one stripe. When $u > 1$, there exists an integer $v \in \left(0, \frac{1}{2}u\right)$ such that one of the following cases happens*:

(i) *The set $\{x_i^b \mid i \le 2v\} \cup \{x_i^t \mid i > u - 2v\}$ contains all rainbow points.*
(ii) *The set $\{x_i^t \mid i \le 2v\} \cup \{x_i^b \mid i > u - 2v\}$ contains all rainbow points.*

Proof The algebraic intersection number of β_1 and e_b is odd. Hence u is also odd. If $u = 1$, then the argument is clear.

Suppose $u > 1$; we show the last argument in three steps. Firstly, if both x_i^b and x_j^b are boundary points of the same rainbow R, then x_k^b for $i < k < j$ are all rainbow

points, otherwise the stripe corresponding to the stripe point x_k^b would intersect R. Thus, rainbow points on e_b are consecutive. The same assertion holds for x_i^t.

Secondly, one of x_1^b and x_1^t must be a rainbow point. Indeed, if this were not true then both x_1^b and x_1^t would be stripe points. They cannot be boundary points of the same stripe, otherwise β_1 would not be connected. They cannot be boundary points of different stripes, otherwise two corresponding stripes would intersect each other. Thus, the assumption is false. Similarly, one of x_u^b and x_u^t must be a rainbow point.

Finally, if x_1^b is a rainbow point then x_u^b cannot be a rainbow point, otherwise all points would be rainbow points. As discussed above, the point x_u^t is a rainbow point. Since the number of rainbow points on e_t is even, there exists an integer v satisfying case (i). If x_1^t is a rainbow point, similar argument implies there exists v satisfying case (ii). □

When $u = 1$, after isotoping β_1, suppose the unique stripe is a vertical line in $C - \{z, w\}$. By moving z through the left edge or the right edge if necessary, suppose basepoints z and w are in different sides of the stripe. If z is on the left of the stripe, set $v = 0$. If z is on the right of the stripe, set $v = 1$.

Then suppose $u > 1$. When in case (i) of Lemma 3.1, rainbows on e_b connect x_i^b to x_{2v+1-i}^b for $i \in [1, v]$, rainbows on e_t connect x_{u+1-i}^t to x_{u-2v+i}^t for $i \in [1, v]$, and stripes connect x_j^b to x_{u+1-j}^t for $j \in [2v + 1, u]$. When in case (ii) of Lemma 3.1, the setting is obtained by replacing i and j by $u + 1 - i$ and $u + 1 - j$, respectively. Without loss of generality, suppose z is in D_1, and w is in D_l. Note that now basepoints cannot be moved through vertical edges of C. Otherwise the rainbows would intersect the vertical edges, which contradicts the definition of the constrained knot. Then we parametrize constrained knots in $L(p, q')$ by the tuple (l, u, v) for case (i) and $(l, u, u - v)$ for case (ii). Since β_1 is connected, we have $\gcd(u, v) = 1$. In summary, the following theorem holds:

Theorem 3.2 *Constrained knots are parametrized by five integers (p, q, l, u, v), where $p > 0$, $q \in [1, p - 1]$, $l \in [1, p]$, $u > 0$, $v \in [0, u - 1]$, $\gcd(p, q) = \gcd(u, v) = 1$ and u is odd. Moreover, $v \in [1, u - 1]$ when $u > 1$ and $v \in \{0, 1\}$ when $u = 1$.*

Note that the parameter v in Theorem 3.2 is different from the integer v in case (ii) of Lemma 3.1. Intuitively, for $v \in [1, u - 1]$ in the parametrization (p, q, l, u, v) with $u > 1$, the number $\min\{v, u - v\}$ is the number of rainbows around a basepoint.

For parameters (p, q, l, u, v), let $C(p, q, l, u, v)$ denote the corresponding constrained knot. When considering the orientation, let $C(p, q, l, u, v)^+$ denote the knot induced by

$(T, \alpha_1, \beta_1, z, w)$ and let $C(p, q, l, u, v)^-$ denote the knot induced by $(T, \alpha_1, \beta_1, w, z)$. For $q \notin [1, p-1]$ and $l \notin [1, p]$, consider the integers q and l modulo p. If $u > 1$ and $v \notin [1, u-1]$, consider the integer v modulo u. For $p < 0$, let $C(p, q, l, u, v)$ denote $C(-p, -q, l, u, v)$.

Remark 3.3 The knot $C(p, q, l, u, v)$ is in $L(p, q')$, where $qq' \equiv 1 \pmod{p}$. Even though $L(p, q)$ is diffeomorphic to $L(p, q')$, constrained knots $C(p, q, l, u, v)$ and $C(p, q', l, u, v)$ are not necessarily equivalent. For example, Theorem 1.2 implies that constrained knots $C(5, 2, 3, 3, 1)$ and $C(5, 3, 3, 3, 1)$ are not equivalent.

We now provide some basic propositions of constrained knots. Also, we indicate the relationship of constrained knots with other families of knots mentioned in Section 2.

Proposition 3.4 $C(p, -q, l, u, -v)$ *is the mirror image of* $C(p, q, l, u, v)$ *for* $u > 1$. $C(p, -q, l, 1, 1)$ *is the mirror image of* $C(p, q, l, 1, 0)$.

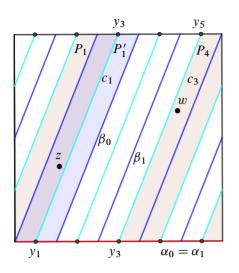**Proof** This follows from the vertical reflection of the standard diagram. □

Hence we only consider $C(p, q, l, u, v)$ with $0 \le 2v < u$ in the rest of the paper.

Proposition 3.5 $$C(1, 0, 1, u, v) \cong \mathfrak{b}(u, v).$$

Proof By cutting along α_1 and a small circle around x in Figure 3, right, the doubly pointed diagram of a 2–bridge knot can be shaped into a square. This proposition is clear by comparing this diagram with the new diagram C related to $C(1, 0, 1, u, v)$. □

Proposition 3.6 *For any fixed orientations of* α_1 *and* β_1 *in the standard diagram of a constrained knot, intersection points* x_i^b *have alternating signs and adjacent strands of* β_1 *in the new diagram* C *have opposite orientations.*

Proof From a similar observation in the proof of Proposition 3.5, for $C(p, q, l, u, v)$, the curve β_1 in the new diagram C is same as the curve β in the doubly pointed Heegaard diagram of $\mathfrak{b}(u, v)$. Thus, it suffices to consider the 2–bridge knot $\mathfrak{b}(u, v)$. The Schubert normal form of $\mathfrak{b}(u, v)$ is the union of two dotted horizontal arcs behind the plane and two winding arcs on the plane. Suppose γ is one of the winding arcs. Then $\beta_1 = \partial N(\gamma)$ cuts the plane into two regions, the inside region $\text{int } N(\gamma)$ and the outside region $\mathbb{R}^2 - N(\gamma)$. Points x and y in Figure 3, right, are in different regions
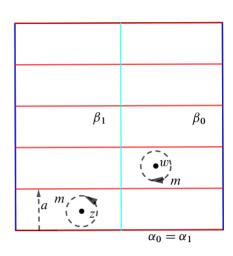
Figure 5: $S(5,2,3) \cong C(5,3,2,1,0)^+$, where regions P_1, P_4 and P_1' are indicated by shadow.

and points x_i^b are on the arc connecting x to y. Since regions on different sides of β_1 must be different, the arc connecting x to y is cut by x_i^b into pieces that lie in the inside region and the outside region alternately. For each piece of the arc, the endpoints are boundary points of a connected arc in β_1. Thus, signs of x_i^b are alternating. The orientations on strands of β_1 are induced by signs of x_i^b. Hence adjacent strands of β_1 have opposite orientations. $\square$

Proposition 3.7 *For $p, q, q' \in \mathbb{Z}$ satisfying $qq' \equiv 1 \pmod{p}$, there are relations*

(i) $S(p, q', k) \cong C(p, q, l, 1, 0)^+$, *where $k - 1 \equiv (l-1)q' \pmod{p}$,*

(ii) $S(p, q', k) \cong C(p, q, l, 1, 1)^+$, *where $k + 1 \equiv (l-1)q' \pmod{p}$.*

Proof Consider curves $\alpha = \alpha_0 = \alpha_1$, β_0 and β_1 in the definition of a constrained knot. When $u = 1$, the curve β_1 is parallel to β_0. Consider the new diagram C and regions D_i for $i \in \mathbb{Z}/p\mathbb{Z}$ as in Figure 4, right. Suppose components of $T^2 - \alpha \cup \beta_1$ are P_i and components of $T^2 - \alpha \cup \beta_0$ are P_i', ordered from left to right as in Figure 5 so that $z \in P_1 \cap P_1'$. Suppose y_i are intersection points of α and β_1 on the bottom edge of P_i'. The strand $c_i = \beta_1 \cap P_i'$ connects y_i to $y_{i+q'}$, so the strand $\beta_1 \cap D_l$ in the new diagram C is $c_{1+(l-1)q'}$. When $v = 0$, the other basepoint w is in $P_{(l-1)q'+2}$, so $k \equiv (l-1)q' + 1 \pmod{p}$. When $v = 1$, the other basepoint w is in $P_{(l-1)q'}$, so $k \equiv (l-1)q' - 1 \pmod{p}$. $\square$

Corollary 3.8 *For $p, q, q' \in \mathbb{Z}$ satisfying $qq' \equiv 1 \pmod{p}$, there are relations*

 (i) $C(p, q, l, 1, 0) \cong C(p, q, l + 2q, 1, 1)$,

 (ii) $C(p, q, l, 1, 0)^+ \cong C(p, q, -2q + 2 - l, 1, 0)^-$,

 (iii) $C(p, q, l, 1, 0)$ *is the mirror image of* $C(p, -q, l + 2q, 1, 0) \cong C(p, -q, l, 1, 1)$,

 (iv) $C(p, q, l, 1, 0) \cong C(p, q', q'l - 2q' + 2, 1, 0)$,

 (v) $S(p, q, k) \cong S(p, q', kq') \cong C(p, q, k - q + 1, 1, 0)^+$.

Proof These relations follow from Propositions 2.7 and 3.7. □

Corollary 3.9 *The knot $C(p, q, -q + 1, 1, 0)$ is an unknot in a lens space. The knot $C(p, q, l, 1, 0)$ for $l = 1, -2q + 1, -q + 2$ or $-q$ is a core knot of a lens space.*

Proof The unknot case is obtained by substituting $k = 0$ in case (v) of Corollary 3.8. Note that $S(p, q, 0)$ is the unknot: the knot is isotopic to a circle bounding a disk on T^2. The core knot cases are obtained by substituting $k = \pm 1, \pm q$ in case (v) of Corollary 3.8. Note that $S(p, q, q)$ is isotopic to a simple closed curve on T^2 that intersects α once, which also is isotopic to the core curve of the α–handlebody. By Proposition 3.7, simple knots $S(p, q, \pm q)$ and $S(p, q, \pm 1)$ are also core knots. □

Proposition 3.10 *For $K = C(p, q, l, 1, 0)$, we have a presentation of the homology*

$$H_1(E(K); \mathbb{Z}) \cong \langle [a], [m] \rangle / (p[a] + k[m]) \cong \mathbb{Z} \oplus \mathbb{Z}/\gcd(p, k)\mathbb{Z},$$

where m is the circle in Figure 5, a is the core curve of α_0–handle and $k \in (0, p]$ satisfies $k - 1 \equiv (l - 1)q^{-1} \pmod{p}$.

Proof This follows from Proposition 3.7 and results in [38, Section 3.3]. □

Proposition 3.11 *Suppose $C(p, q, l, u, v)$ is a constrained knot in $L(p, q')$ with $0 \le 2v < u$. Let $q_i \in [0, p)$ be integers satisfying $q_i \equiv iq' \pmod{p}$ and let $k \in [1, p]$ be the integer satisfying $k - 1 \equiv (l - 1)q' \pmod{p}$. Moreover, let*

$$n_1 = \#\{i \in [0, l - 1] \mid q_i \in [0, k - 1]\} \quad \text{and} \quad n_2 = \#\{i \in [0, l - 1] \mid q_i \in [1, q' - 1]\}.$$

Then $C(p, q, l, u, v) \cong W(pu - 2v(l - 1), v, uk - 2vn_1, uq' - 2vn_2)$.

Proof The parameters $(p - l + 1)u + (l - 1)(u - 2v) = pu - 2v(l - 1)$ and v are from counting the numbers of intersection points and rainbows in the standard

diagram of a constrained knot, respectively. Suppose that P_i' are components of $T^2 - \alpha_0 \cup \beta_0$ in the standard diagram of $L(p, q')$, ordered from left to right so that $z \in P_1'$. Similar to the proof of Proposition 3.7, we know $w \in P_k'$. Then the parameter $(k - n_1)u + n_1(u - 2v) = uk - 2vn_1$ counts the number of stripes between rainbows and the parameter $(q' - n_2)u + n_2(u - 2v) = uq' - 2vn_2$ counts the twisting number. □

Proof of Theorem 1.11 For a knot K in a lens space with Seifert fibered complement, any Dehn surgery other than the one along homological longitude gives a Seifert fibered space. By discussion in [39, Section 5], all oriented Seifert fibered spaces over $\mathbb{RP}^2$ are L-spaces and the classification of L-spaces over S^2 is given by [39, Theorem 5.1]. Moreover, no higher genus Seifert fibered spaces are L-spaces. The classification in [39, Theorem 5.1] indicates there are at least two Dehn fillings on the knot complement that are L-spaces, ie K is always an L-space knot. By Proposition 3.11, we can transform standard diagrams of constrained knots into $(1, 1)$ diagrams. By Proposition 3.6 and Theorem 2.5, a constrained knot is an L-space knot if and only if $(u, v) = (1, 0)$ or $(1, 1)$, or $u > 1$ and $v = \pm 1$.   □

Proof of Proposition 1.1 The necessary part of the proposition follows directly from the definition of constrained knots: the intersection points of α_1 and β_1 between two consecutive intersection points of α_0 and β_0 correspond the same spin^c structure on the lens space, where α_1, β_1, α_0 and β_0 are curves in the standard diagrams of the constrained knot and the lens space.

We prove the sufficient part of this proposition. For simplicity, intervals are considered in $\mathbb{Z}/p\mathbb{Z}$. In particular, let $(p_k, p_1]$ denote $(0, p_1] \cup (p_k, p]$. Consider intersection points x_i for $i \in [1, p]$ as shown in Figure 1, right.

Firstly, spin^c structures $\mathfrak{s}_i$ are equal for all $i \in [r + 1, r + 2q]$. Indeed, for $i \in [1, q]$, the points x_{r+i} and $x_{r+2q+1-i}$ are boundary points of a rainbow, that is there is a holomorphic disk connecting x_{r+i} to $x_{r+2q+1-i}$. Thus $\mathfrak{s}_{r+i} = \mathfrak{s}_{r+2q+1-i}$. If $q = 1$, this assertion is trivial. If $q > 1$ and the assertion did not hold, then there must be an integer q_0 and two spin^c structures $\mathfrak{s}_A$ and $\mathfrak{s}_B$ such that $\mathfrak{s}_i = \mathfrak{s}_A$ for all $i \in [r + q_0, r + 2q + 1 - q_0]$ and $\mathfrak{s}_j = \mathfrak{s}_B$ for all $j \notin [r + q_0, r + 2q + 1 - q_0]$, which implies $a = 2$. Since spin^c structures of two boundary points of a stripe are different, for all $i \in [2q + 1 - s, p - s]$, spin^c structures $\mathfrak{s}_i$ are different from $\mathfrak{s}_B$. Thus $\mathfrak{s}_i = \mathfrak{s}_A$ for all $i \in [2q + 1 - s, p - s]$. For $i \in [1, q]$, points x_{i-s} and $x_{2q+1-i-s}$ are boundary points of a rainbow, so $\mathfrak{s}_{i-s} = \mathfrak{s}_{2q+1-i-s}$. Since there are $2q_0$ points corresponding

to $\mathfrak{s}_A$, integers q_0 should satisfy the inequality $2q_0 > p - 2q$. For $i \in \left[q + q_0 - \frac{1}{2} p, q \right]$, points x_{i-s} and $x_{2q+1-i-s}$ correspond to $\mathfrak{s}_B$. In particular, points x_{r+1} and x_{r+2q} are identified with x_{2q+1-i_0-s} and x_{i_0-s} for $i_0 = q + q_0 - \frac{1}{2} p$, respectively. Let R_1 be the rainbow with boundary points x_{r+1} and x_{r+2q}, and let R_2 be the rainbow with boundary points x_{2q+1-i_0-s} and x_{i_0-s}. The union of R_1 and R_2 becomes a component of β, which contradicts the assumption that β only has one component.

We can similarly show that the spin^c structures $\mathfrak{s}_i$ are equal for all $i \in [1 - s, 2q - s]$. From this discussion, for any $i \in [1, k]$, we have

$$ p_i \neq r + 1, r + 2, \dots, r + 2q - 1, 1 - s, 2 - s, \dots, 2q - 1 - s. $$

Suppose y_i for $i \in [1, k]$ are points on α between x_{p_i} and x_{p_i+1}. If $p_i \neq r, r + 2q$ or p, then p_i and $p_i + 1$ must be boundary points of two successive stripes. Suppose x_j and x_{j+1} are the other boundary points of these stripes, respectively. There must be a point y_j between x_j and x_{j+1} because $\mathfrak{s}_j - \mathfrak{s}_{j+1} = \mathfrak{s}_{p_i} - \mathfrak{s}_{p_i+1} \neq 0$. Let b_i be a strand connecting y_i to y_j which is disjoint from β.

Suppose $p_i = p$. If $r \neq 0$ and $p - 2q - r \neq 0$ there are stripes connecting $\mathfrak{s}_p$ to $\mathfrak{s}_{p-s}$ and connecting $\mathfrak{s}_1$ to $\mathfrak{s}_{2q+1-s}$, respectively. Thus $\mathfrak{s}_{p-s} - \mathfrak{s}_{2q+1-s} = \mathfrak{s}_p - \mathfrak{s}_1 \neq 0$. There is a point y_j either between x_{p-s} and x_{1-s} or between x_{2q-s} and x_{2q+1-s} for some j. Only one case will happen because the number of intersection points corresponding to any fixed spin^c structure is odd. Let b_i be a strand connecting y_i to y_j which is disjoint from β. If either $r = 0$ or $p - 2q - r = 0$, by choosing different stripes, $\mathfrak{s}_{p-s} - \mathfrak{s}_{2q+1-s} \neq 0$ still holds. The point y_j and the strand b_i can also be found. By a similar argument, this is also true for $p_i = r$ and $r + 2q$.

Let β_0 be the union of b_i. Without considering basepoints, β_0 is isotopic to β. Thus, it has only one component. Finally, the curves β_0, α and β can be identified with β_0, α_1 and β_1 in the definition of a constrained knot. Thus, we conclude that the (1,1) knot is a constrained knot. $\qquad\square$

4 Knot Floer homology

Heegaard Floer homology is an invariant for closed 3–manifolds discovered by Ozsváth and Szabó [30; 31]. It has been generalized to knot Flor homology [29; 36], sutured Floer homology [20], bordered Floer homology [22] and immersed curves for manifolds

with torus boundary [16; 17]. See [38, Section 3] for a brief review of knot Floer homology for rationally null-homologous knots. See also [33].

In this section suppose $K = C(p, q, l, u, v)$ is a constrained knot in $Y = L(p, q')$, where $qq' \equiv 1 \pmod{p}$. Write $H_1 = H_1(E(K); \mathbb{Z})$ and $\widehat{HFK}(K) = \widehat{HFK}(Y, K)$ for short. For any homogeneous element $x \in \widehat{HFK}(K)$, let $\mathrm{gr}(x) \in H_1$ be the Alexander grading of x mentioned in the introduction. Note that the Alexander grading is well-defined up to a global grading shift [11], ie up to multiplication by an element in H_1. However, the difference $\mathrm{gr}(x) - \mathrm{gr}(y)$ for two homogeneous elements x and y is always well-defined. This difference can be calculated explicitly by the doubly pointed Heegaard diagram of the knot by the approach in [38, Section 3.3].

Consider the group ring $\mathbb{Z}[H_1]$. Two elements f_1 and f_2 in $\mathbb{Z}[H_1]$ are *equivalent*, denoted by $f_1 \sim f_2$, if there exists an element $g \in \pm H_1$ such that $f_1 = g f_2$. For any element $h \in H_1$, there is a grading summand $\widehat{HFK}(K, h)$ of $\widehat{HFK}(K)$ as in (2). There is also a relative $\mathbb{Z}/2$ grading on $\widehat{HFK}(K)$ induced by signs of the intersection numbers in the Heegaard diagram (see [11, Section 2.4]) and related to the modulo 2 Maslov grading on $\widehat{HFK}(K, \mathfrak{s})$. This grading respects the Alexander grading and induces a $\mathbb{Z}/2$ grading on $\widehat{HFK}(K, h)$. Then the Euler characteristic $\chi(\widehat{HFK}(K, h))$ is well-defined up to sign. We can consider the (graded) Euler characteristic of $\widehat{HFK}(K)$:

$$\chi(\widehat{HFK}(K)) = \sum_{h \in H_1} \chi(\widehat{HFK}(K, h)) \cdot h$$
$$= \sum_{h \in H_1} (\mathrm{rk}\, \widehat{HFK}_{\mathrm{even}}(K, h) - \mathrm{rk}\, \widehat{HFK}_{\mathrm{odd}}(K, h)) \cdot h.$$

From the above discussion, we know $\chi(\widehat{HFK}(K))$ is an element in $\mathbb{Z}[H_1]$ up to equivalence. For $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, we consider $\widehat{HFK}(K, \mathfrak{s})$ as a subgroup of $\widehat{HFK}(K)$ so that it also has an H_1–grading and $\chi(\widehat{HFK}(K, \mathfrak{s}))$ is also an element in $\mathbb{Z}[H_1]$ up to equivalence.

For a constrained knot K, we will show $\widehat{HFK}(K)$ totally depends on $\chi(\widehat{HFK}(K))$. Explicitly this means that, for any $h \in H_1$, the dimension of $\widehat{HFK}(K, h)$ is the same as the absolute value $|\chi(\widehat{HFK}(K, h))|$.

As shown in Figures 4 and 6, suppose e^j is the top edge of D_j and x_i^j is the intersection point of e^j and β_1 for $j \in \mathbb{Z}/p\mathbb{Z}$ and $i \in [1, u(j)]$. Let $x_{\mathrm{middle}}^j = x_{(u(j)+1)/2}^j$ be middle points. It is clear that $\mathfrak{s}_z(x_{i_1}^{j_1}) = \mathfrak{s}_z(x_{i_2}^{j_2})$ if and only if $j_1 = j_2$. For any integer $j \in [1, p]$, define $\mathfrak{s}_j = \mathfrak{s}_z(x_{\mathrm{middle}}^j) \in \mathrm{Spin}^c(Y)$.
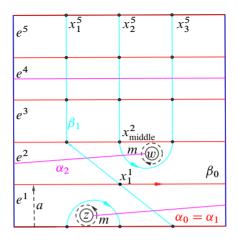
Figure 6: Heegaard diagram of $E(C(5,3,2,3,1))$.

Lemma 4.1 For $K = C(p,q,l,u,v)$ with $u > 2v > 0$, suppose $k \in (0, p]$ is the integer satisfying $k - 1 \equiv (l-1)q^{-1} \pmod{p}$. Define

$$k' = \begin{cases} k - 2 & \text{if } v \text{ is odd,} \\ k & \text{if } v \text{ is even.} \end{cases}$$

Suppose $d = \gcd(p, k')$. Then there is a presentation of the homology H_1:

$$H_1 = H_1(E(K); \mathbb{Z}) \cong \langle [a], [m] \rangle / (p[a] + k'[m]) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z},$$

where m is the circle in Figure 6 and a is the core curve of α_0–handle.

Proof Suppose β_1 is oriented so that the orientation of the middle stripe is from bottom to top. Let $[\beta_1(p,q,l,u,v)]$ denote the homology class of β_1 corresponding to $C(p,q,l,u,v)$. By Proposition 3.6, orientations of rainbows around a basepoint are alternating. Note that moving all rainbows of β_1 across basepoints gives the diagram of the simple knot $C(p,q,l,1,0)$. Then

$$[\beta_1(p,q,l,u,v)] + 2[m] = [\beta_1(p,q,l,1,0)] \quad \text{if } v \text{ is odd,}$$
$$[\beta_1(p,q,l,u,v)] = [\beta_1(p,q,l,1,0)] \quad \text{if } v \text{ is even.}$$

Then this proposition follows from Proposition 3.10. Note that $[a]$ and $[m]$ correspond to core curves of α_1 and α_2, and the relation in the presentation of H_1 corresponds to algebraic intersection numbers $\alpha_1 \cdot \beta$ and $\alpha_2 \cdot \beta$; see Section 6 for the approach to obtain a presentation of $\pi_1(E(K))$ and note that H_1 is the abelianization of $\pi_1(E(K))$. $\square$

Lemma 4.2 ([Proposition 1.4](#)) *For $K = C(p, q, l, u, v)$ with $u > 2v \geq 0$, suppose H_1 is presented as in [Lemma 4.1](#). For any integer $j \in [1, p]$, let $\mathfrak{s}_j = \mathfrak{s}_z(x_{\mathrm{middle}}^j)$ for intersection points x_{middle}^j in [Figure 6](#). Then for any j, the group $\widehat{HFK}(K, \mathfrak{s}_j)$ is determined by its Euler characteristic.*

Moreover, suppose integers u' and v' satisfy $u' = u - 2v$ and $v' \equiv v \pmod{u'}$. Let $\Delta_1(t)$ and $\Delta_2(t)$ be Alexander polynomials of $\mathfrak{b}(u, v)$ and $\mathfrak{b}(u', v')$, respectively. Then

$$\chi(\widehat{HFK}(K, \mathfrak{s}_j)) \sim \begin{cases} \Delta_1([m]) & \text{if } j \in [l, p], \\ \Delta_2([m]) & \text{if } j \in [1, l-1]. \end{cases}$$

Proof For $j \in [1, p]$, consider the edge e^j and the intersection numbers x_i^j of e^j and β_1 in the diagram C. Suppose $(e^j)'$ is the curve obtained by identifying two endpoints of e^j. For $j \in [l, p]$, the diagram $(T^2, (e^j)', \beta_1, z, w)$ is the same as the diagram of $K_1 = \mathfrak{b}(u, v)$. For $j \in [1, l-1]$, by case (iii) of [Lemma 7.8](#), the diagram $(T^2, (e^j)', \beta_1, z, w)$ is isotopic to the diagram of $K_2 = \mathfrak{b}(u', v')$. For the readers' convenience, we sketch the proof.

The fact that $u' = u - 2v$ follows directly from the number of intersection points of $(e^j)'$ and β_1, which is the same as the number of stripes. Then we consider v'. Let $D = N(x_{\mathrm{middle}}^p)$ be a neighborhood of x_{middle}^p such that D contains all rainbows. Consider the isotopy obtained by rotating D counterclockwise. If $v > u'$, after rotation the resulting diagram has $v - u'$ rainbows. The formula for v' follows by induction.

Since 2–bridge knots are alternating they are thin [28] in the sense of [Definition 1.3](#). By comparing the number of generators of $\widehat{CFK}(K_i)$ for $i = 1, 2$ from $(T^2, (e^j)', \beta_1, z, w)$ and the dimension of $\widehat{HFK}(K_i)$ from the Alexander polynomial (see [Proposition 2.9](#)), we know there is no differential on $\widehat{CFK}(K_i)$. This fact can also be shown by a direct calculation following the method in [14]. Thus, the constrained knot K is also thin and there is no differential on $\widehat{CFK}(K, \mathfrak{s}_j)$. In particular, the group $\widehat{CFK}(K, \mathfrak{s}_j)$ is determined by its Euler characteristic.

As discussed at the start of this section, the characteristic $\chi(\widehat{HFK}(K, \mathfrak{s}_j))$ is an element in H_1 up to equivalence. Similar to the proof of [35, Lemma 3.4], for $j \in [l, p]$,

$$\mathrm{gr}(x_{i+1}^j) - \mathrm{gr}(x_i^j) = [m]^{(-1)^{\lfloor iv/u \rfloor}}.$$

For $j \in [1, l-1]$, just replace u and v by u' and v' in the above formula, respectively. Comparing the formula for the Alexander polynomial in [Proposition 2.9](#), we conclude the formula for $\chi(\widehat{HFK}(K, \mathfrak{s}_j))$. $\qquad\square$

Lemma 4.3 *Consider integers k and k', and the presentation of H_1 as in Lemma 4.1. For $j \neq 0, l-1$,*

$$\text{gr}(x_{\text{middle}}^{j+1}) - \text{gr}(x_{\text{middle}}^{j}) = \begin{cases} [a] + [m] & \text{if } jq^{-1} \equiv 1, \dots, k-2 \pmod{p}, \\ [a] & \text{otherwise.} \end{cases}$$

For $l \neq 1$ and $j = 0, l-1$,

$$\text{gr}(x_{\text{middle}}^{j+1}) - \text{gr}(x_{\text{middle}}^{j}) = \begin{cases} [a] + [m] & \text{if } v \text{ is even,} \\ [a] & \text{if } v \text{ is odd.} \end{cases}$$

For $l = 1$,

$$\text{gr}(x_{\text{middle}}^{j+1}) - \text{gr}(x_{\text{middle}}^{j}) = \begin{cases} [a] + [m] & \text{if } v \text{ is even,} \\ [a] - [m] & \text{if } v \text{ is odd.} \end{cases}$$

Proof For simple knots, the proof is based on Fox calculus; see [38, Proposition 6.1]. For a general constrained knot and $j \neq 0, l-1$, the proof in [38] still works because orientations of strands are alternating. The differences of gradings for $j = 0$ and $j = l-1$ are the same because z and w are symmetric by rotation. The proof follows from

$$\sum_{j=0}^{p-1} \text{gr}(x_{\text{middle}}^{j+1}) - \text{gr}(x_{\text{middle}}^{j}) = 0 \in H_1 \quad \text{and} \quad p[a] + k'[m] = 0 \in H_1. \qquad \square$$

Corollary 4.4 *Suppose $K = C(p, q, l, u, v)$ is a constrained knot in $Y = L(p, q')$, where $qq' \equiv 1 \pmod{p}$. For any integer $j \in [1, p]$, let $\mathfrak{s}_j = \mathfrak{s}_z(x_{\text{middle}}^j) \in \text{Spin}^c(Y)$ for intersection points x_{middle}^j in Figure 6. Then $\mathfrak{s}_{j+1} - \mathfrak{s}_j$ only depends on p and q.*

Proof By the map

$$H_1(E(K); \mathbb{Z})/([m]) \to H_1(Y; \mathbb{Z}),$$

the grading difference $\text{gr}(x_{\text{middle}}^{j+1}) - \text{gr}(x_{\text{middle}}^{j})$ is mapped to $\mathfrak{s}_{j+1} - \mathfrak{s}_j$, which only depends on the image of $[a]$. $\qquad \square$

Lemma 4.5 *Consider $\mathfrak{b}(u, v)$ and $\mathfrak{b}(u', v')$ as in Lemma 4.2. Then*

$$\sigma(\mathfrak{b}(u', v')) = \begin{cases} \sigma(\mathfrak{b}(u, v)) & \text{if } v \text{ is even,} \\ \sigma(\mathfrak{b}(u, v)) + 2 & \text{if } v \text{ is odd.} \end{cases}$$

Proof Consider standard presentations of 2–bridge knots in Section 2.2. It is easy to see $\mathfrak{b}(u, v)$ and $\mathfrak{b}(u', v')$ form two knots in the skein relation. By the skein relation formula of signatures of knots, we can conclude this lemma. Moreover, we provide another proof based on the Alexander grading.

By the algorithm of the Alexander grading, we have

$$\mathrm{gr}(x_{u'}^1) - \mathrm{gr}(x_u^0) = [a] + [m].$$

From the rotation symmetry and the formula of the signature in Proposition 2.9,

$$\mathrm{gr}(x_u^0) - \mathrm{gr}(x_{\mathrm{middle}}^0) = \mathrm{gr}(x_{\mathrm{middle}}^0) - \mathrm{gr}(x_1^0) = \tfrac{1}{2}\sigma(\mathfrak{b}(u,v))[m],$$

$$\mathrm{gr}(x_{u'}^1) - \mathrm{gr}(x_{\mathrm{middle}}^1) = \mathrm{gr}(x_{\mathrm{middle}}^1) - \mathrm{gr}(x_1^1) = \tfrac{1}{2}\sigma(\mathfrak{b}(u',v'))[m].$$

Then this lemma follows from these equations and Lemma 4.3.   □

Theorem 4.6 *For a constrained knot* $K = C(p,q,l,u,v)$, *consider the Alexander polynomials* $\Delta_1(t)$ *and* $\Delta_2(t)$ *in Lemma 4.2. Then* $\widehat{HFK}(K)$ *is determined by its Euler characteristic, which is calculated by*

$$(3) \qquad \chi(\widehat{HFK}(K)) = \Delta_1([m]) \sum_{j=l}^{p} \mathrm{gr}(x_{\mathrm{middle}}^j) + \Delta_2([m]) \sum_{j=1}^{l-1} \mathrm{gr}(x_{\mathrm{middle}}^j).$$

Proof By the result of Lemma 4.2, we only need to consider the (relative) signs of intersection points corresponding to different spin^c structures. By Proposition 3.6, signs of intersection points x_i^j for fixed j are alternating. Since u and $u' = u - 2v$ are odd, signs of x_1^j and $x_{u(j)}^j$ are the same, where $u(j)$ is either u or u' by Lemma 4.2. From the diagram, signs of $x_{u(j)}^j$ for $j \in [0,l]$ are the same and signs of x_1^k for $k \in [l,p]$ are the same. Thus, we obtain (3).   □

All terms in (3) can be calculated by Lemmas 4.3 and 4.5. Thus, we obtain an algorithm for $\widehat{HFK}(K)$ for a constrained knot K.

Let signs of x_1^j be positive. The Alexander grading can be fixed by the global symmetry, ie we consider the absolute Alexander grading. Note that the global symmetry corresponds to switching the roles of z and w, which is equivalent to a rotation of the standard diagram of a constrained knot. Then we have

$$\mathrm{gr}(x_{\mathrm{middle}}^j) = -\mathrm{gr}(x_{\mathrm{middle}}^{2l-j}) \text{ for any } j.$$

In this assumption we may use square roots of elements in H_1 to achieve the symmetry, and the Euler characteristic $\chi(\widehat{HFK}(K))$ is a well-defined element in $(\tfrac{1}{2}\mathbb{Z})[H_1]$ for this case. The group $\widehat{HFK}(K)$ with the Alexander grading fixed as above is called the *canonical representative*.

Proof of the necessary part of Theorem 1.2 For $i = 1, 2$, if $K_i = C(p_i, q_i, l_i, u_i, v_i)$ are equivalent, then $p_1 = p_2 = p$ and $q_1 \equiv q_2^{\pm 1} \pmod{p}$ by the classification of lens spaces [5]. Suppose Y is the lens space containing K_1 and K_2. For $i = 1, 2$, consider (u_i', v_i') as in Lemma 4.2. By comparing knot Floer homologies, we know $l_1 = l_2$ and

$$u_1 = |\Delta_\mathfrak{b}(u_1, v_1)(-1)| = |\Delta_\mathfrak{b}(u_2, v_2)(-1)| = u_2,$$

$$u_1 - 2v_1 = |\Delta_{\mathfrak{b}(u_1', v_1')}(-1)| = |\Delta_{\mathfrak{b}(u_2', v_2')}(-1)| = u_2 - 2v_2.$$

Thus, we have $(l_1, u_1, v_1) = (l_2, u_2, v_2) = (l, u, v)$. Moreover, the sets of spin^c structures corresponding to $\mathfrak{b}(u, v)$ for two constrained knots should be the same. By Corollary 4.4, it suffices to consider simple knots. Let $\mathfrak{s}_j^i$ be spin^c structures related to diagrams of K_i for $i = 1, 2$. Traveling along α_1 of K_1, middle points are in the order

$$x_{\text{middle}}^0, x_{\text{middle}}^{q_1}, \ldots, x_{\text{middle}}^{(p-1)q_1}.$$

Thus, we have

$$\mathfrak{s}_{q_1+j}^1 - \mathfrak{s}_j^1 = \mathfrak{s}_{j+1}^2 - \mathfrak{s}_j^2 \in H^2(Y; \mathbb{Z}).$$

Then the following sets are the same:

$$\{\mathfrak{s}_j^1 - \mathfrak{s}_0^1 + \mathfrak{s}_j^1 - \mathfrak{s}_1^1 \mid j \in [l, p]\}, \quad \{\mathfrak{s}_j^2 - \mathfrak{s}_0^2 + \mathfrak{s}_j^2 - \mathfrak{s}_1^2 \mid j \in [l, p]\}.$$

Equivalently, numbers in $\{0, q_1, \ldots, (p - l)q_1\}$ should be consecutive congruence classes modulo p. By the following proposition, this can only happen when $l \in \{2, p\}$. □

Proposition 4.7 *Suppose that integers p, q and k satisfy $1 < q < p - 1$, $\gcd(p, q) = 1$ and $0 \le k < p - 1$. Then there exists an integer x such that the sets $\{x, x + 1, \ldots, x + k\}$ and $\{0, q, \ldots, kq\}$ can be identified modulo p if and only if $k = 0, p - 2$.*

Proof If $k = 0, p - 2$, this proposition is trivial. Suppose $k \ne 0, p - 2$. Assume elements in sets are in $\mathbb{Z}/p\mathbb{Z}$ in this proof. Define

$$T = \{0, 1, \ldots, p - 1\}, \quad S^q = \{0, q, \ldots, kq\} \quad \text{and} \quad S_x = \{x, x + 1, \ldots, x + k\}.$$

Suppose $S^q = S_x$ for some x and $n = \lfloor p/q \rfloor \ge 2$. If $k \le n$, then the set S^q cannot be identified with S_x. Thus $k \ge n + 1$ and $\{0, q, \ldots, nq\} \subset S^q = S_x$. Suppose $T - S_x = \{y, y+1, \ldots, y+p-k-2\}$, where $y = x+k+1$. Since $(T - S_x) \cap S^q$ is empty by assumption, the set $T - S_x$ must be either a subset of $\{iq + 1, iq + 2, \ldots, (i+1)q - 1\}$ for some integer $i \in [0, n - 1]$ or a subset of $\{nq + 1, nq + 2, \ldots, p - 1\}$. If $q = 2$, then $k = 0$, which contradicts the assumption. Suppose $q > 2$. Since $k \ne 0, p - 2$, we know $y, y + 1 \in T - S_x$.

If the first case happens with $i = 0$, then we know $\{q+1, q+2, \ldots, 2q-1\} \subset S_x = S_q$ because $n \geq 2$. Since $y+q, y+1+q \in \{q+1, q+2, \ldots, 2q-1\}$, there exist different integers $k_0, k_1 \in [1, k]$ such that

$$y + q \equiv k_0 q \quad \text{and} \quad y + 1 + q \equiv k_1 q \pmod{p}.$$

If $k_0 > k_1$, then $k_0 - 1 \in [1, k-1]$ and $y = (k_0 - 1)q \in S^q$. If $k_0 < k_1$, then $k_1 - 1 \in [1, k-1]$ and $y + 1 = (k_1 - 1)q \in S^q$. Both contradict the assumption.

If the first case happens with $i > 0$ or the second case happens, then there exist different integers $k_0, k_1 \in [1, k]$ such that

$$y - q \equiv k_0 q \quad \text{and} \quad y + 1 - q \equiv k_1 q \pmod{p}.$$

If $k_0 > k_1$, then $k_1 + 1 \in [2, k]$ and $y + 1 = (k_1 + 1)q \in S^q$. If $k_0 < k_1$, then $k_0 + 1 \in [2, k]$ and $y = (k_0 + 1)q \in S^q$. Both contradict the assumption.

In summary, for $p > 2q$, there is a contradiction if $k \neq 0, p-2$. If $p < 2q$ and $S^q = S_x$, then we consider

$$S^{p-q} = \{-x, -x-1, \ldots, -x-k\} = S_{-x-k}.$$

Note that $p > 2(p - q)$. From a similar discussion, there is also a contradiction. □

In the rest of this section, we indicate how to draw the curve invariant [16; 17] of the knot complement of a constrained knot. Readers who are not familiar with the curve invariant can safely skip the following discussion since there is no further result in this paper relying on it.

Suppose that $K = C(p, q, l, u, v)$ is a constrained knot in $Y = L(p, q')$, where $qq' \equiv 1 \pmod{p}$. Let $M = E(K)$. From the standard diagram of the constrained knot, we know $[K] = k'[b] \in H_1(Y; \mathbb{Z})$, where b is the core curve of β_0–handle and k' is the integer in Lemma 4.1. Since K is thin, the curve invariant $\widehat{HF}(M)$ can be drawn as follows.

The curve invariant can be decomposed with respect to $\text{Spin}^c(M)$, which is affine over $H^2(M; \mathbb{Z})$. By Poincaré duality and the long exact sequence for $(M, \partial M)$, we know

$$|H^2(M; \mathbb{Z})| = |H_1(M, \partial M; \mathbb{Z})| = |H_1(M; \mathbb{Z})/\text{Im}(H_1(\partial M; \mathbb{Z}))| = |\text{Tors } H_1(M; \mathbb{Z})|.$$

For simplicity, suppose $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$. Then $|\text{Spin}^c(M)| = 1$ and $\gcd(p, k') = 1$.

The curve invariant can be lifted to the universal cover $\mathbb{R}^2$ of ∂M. Suppose the basis is $([l^*], -[m^*])$, where the homological meridian m^* (see Section 2) is chosen so that
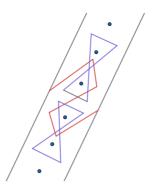
Figure 7: Part of the curve invariant of $C(p,q,l,11,3)$.

$[m] = p[m^*] - k_0[l^*]$ for some $k_0 \in [0, p)$. Consider parallel lines with slope p/k_0 away from the basepoint on M. They cut $\mathbb{R}^2$ into bands. Suppose that lifts of the basepoint are integer points and lie on a line with slope p/k_0 in each band. Since $\widehat{HF}(Y, \mathfrak{s}) \cong \mathbb{Z}$ for any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, the curve invariant intersects each line once.

Based on the proof of Lemma 4.2, the chain complex $\widehat{CFK}(K, \mathfrak{s})$ for any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is similar to the chain complex related to a 2–bridge knot. Moreover, from the relation of the standard diagram of K and the Heegaard diagram of a 2–bridge knot, the minus version of the knot Floer chain complex $CFK^-(K, \mathfrak{s})$ is also related to CFK^- of a 2–bridge knot. From the results in [34, Section 3] about thin complexes and the results in [17, Section 4] about how to draw the curve invariant from CFK^-, the part of the curve invariant of K in a band is the union of some purple figure-8 curves and a distinguished red arc as shown in Figure 7, which totally depends on the Alexander polynomial and the signature of the related 2–bridge knot.

Lemma 4.8 *Suppose $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ and consider k_0 and k' as above. Suppose a and b are core curves of α_0 and β_0 handles corresponding to the standard diagram of $Y = L(p, q')$. Then $k_0 q(k')^2 \equiv -1 \pmod{p}$. Hence k_0 is determined by k'.*

Proof The homology $H_1(M; \mathbb{Z})$ is generated by $[m^*]$. Let $\tilde{m}^*$ denote the image of $[m^*]$ in $H_1(Y; \mathbb{Z})$. By Lemma 4.1, $[a] = -k'\tilde{m}^*$. The relation $[b] = q[a]$ implies $[K] = -q(k')^2 \tilde{m}^*$. Then a lift of $[K]$ in $H_1(T^2; \mathbb{Z})$ equals $-q(k')^2[m^*] + k_1[l^*]$ for some k_1. Since l is isotopic to K, we have $[K] = [l] \in H_1(T^2; \mathbb{Z})$. Then since $[m] = p[m^*] - k_0[l^*]$ and $[m] \cdot [l] = [m^*] \cdot [l^*] = -1$, we have

$$[m] \cdot [l] = (p[m^*] - k_0[l^*]) \cdot (-q(k')^2[m^*] + k_1[l^*]) = (pk_1 - k_0 q(k')^2)[m^*] \cdot [l^*].$$

Hence we conclude the congruence result for k_0. □

For $i \in \mathbb{Z}/p\mathbb{Z}$, suppose B_i are bands in $\mathbb{R}^2$ mentioned above, ordered from left to right. Suppose $\mathfrak{s}_i \in \mathrm{Spin}^c(Y)$ are spin^c structures corresponding to B_i. Since the slope of parallel lines is p/k_0, the difference $\mathfrak{s}_{i+1} - \mathfrak{s}_i$ is $k_0' \tilde{m}^*$ for the integer k_0' satisfying $k_0 k_0' \equiv -1 \pmod{p}$. By the above lemma, we have $k_0' \equiv q(k')^2 \pmod{p}$. By definition of k' in Lemma 4.1, we have

$$-qk' \equiv \begin{cases} -q - l + 1 & \text{if } v \text{ is even modulo } p, \\ q - l + 1 & \text{if } v \text{ is odd modulo } p. \end{cases}$$

Since $[a] = -k'\tilde{m}^*$, bands $B_{-iqk'}$ for $i \in [1, l-1]$ correspond to $\mathfrak{b}(u', v')$ and $B_{-iqk'}$ for $i \in [l, p]$ correspond to $\mathfrak{b}(u, v)$ in $\widehat{HF}(M)$. Finally, the Alexander grading indicates the relative height of the curves in bands and there is a unique way to connect curves in different bands.

5 Knots in the same homology class

For fixed (p, q, u, v) and each $h \in H_1(L(p, q'); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, where $qq' \equiv 1 \pmod{p}$, there is a parameter $l \in [1, p]$ such that $C(p, q, l, u, v)$ is a representative of h, ie $[C(p, q, l, u, v)] = h$. In other words, for any knot K in $L(p, q')$ there are infinitely many constrained knots K' satisfying $[K'] = [K] \in H_1(L(p, q'); \mathbb{Z})$.

In this section we focus on knots representing the same homology class in a lens space. The main results are Theorems 1.5 and 1.6. Since we will not use the parameters of a constrained knot, we denote a lens space by $L(p, q)$ rather than $L(p, q')$ as in other sections. Many results in this section are related to the Turaev torsion $\tau(M)$ of a 3–manifold M with torus boundary [42], which can be calculated by any presentation of $\pi_1(M)$. For simplicity, write $\tau(K) = \tau(E(K))$. The following proposition enables us to compare elements in homology groups of different knot complements:

Proposition 5.1 [5] *Let K be a knot in a 3–manifold Y. The isomorphism class of the homology $H_1(E(K); \mathbb{Z})$ only depends on the homology class $[K] \in H_1(Y; \mathbb{Z})$.*

Suppose $Y = L(p, q)$ and K is a knot in Y. By Proposition 3.10, Lemma 4.1 and Proposition 5.1, there exists a positive integer d satisfying $H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$. Let m be the meridian of K in the sense of Section 2. Suppose t and r are generators of $\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ such that

$$H_1(E(K); \mathbb{Z}) \cong \langle t, r \rangle / (dr).$$

Then there exist $p_0, a \in \mathbb{Z}$ such that the above isomorphism sends $[m]$ to $p_0 t + ar$.

Lemma 5.2 *The integer p is divisible by d, and $p_0 = \pm p/d$. Moreover, the greatest common divisor of p_0, d and a is 1.*

Proof By the isomorphism $H_1(E(K); \mathbb{Z})/([m]) \cong H_1(Y; \mathbb{Z})$, the order p of $H_1(Y; \mathbb{Z})$ is the same as

$$\left| \det\left(\begin{bmatrix} p_0 & a \\ 0 & d \end{bmatrix} \right) \right| = |dp_0|.$$

If the greatest common divisor of p_0, d and a is not 1, then the Smith normal form of this matrix cannot be

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$$

because elementary transformations in the algorithm of the Smith normal form do not decrease the common divisor of all entries. $\square$

Lemma 5.3 *Let K_1 and K_2 be knots in $Y = L(p, q)$ representing the same homology class $h \in H_1(Y; \mathbb{Z})$. Let m_1 and m_2 be meridians of K_1 and K_2 in the sense of [Section 2](). For $i = 1, 2$, there are isomorphisms $j_i \colon H_1(E(K_i); \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ such that $j_1([m_1]) = j_2([m_2])$.*

Proof For $i = 1, 2$, by the discussion after [Proposition 5.1](), there exists an isomorphism $j_i' \colon H_1(E(K_i); \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ such that

$$j_1'([m_1]) = p_0 t + a r \quad \text{and} \quad j_2'([m_2]) = p_0' t + b r.$$

Then it suffices to find an automorphism f of $\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ such that

$$f(p_0 t + a r) = p_0' t + b r.$$

By [Lemma 5.2](), the integers p_0 and p_0' are in $\{p/d, -p/d\}$. Let f_0 be the automorphism of $\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ sending (t, r) to $(-t, r)$. If $p_0 = -p/d$, the map j_1' can be replaced by $f_0 \circ j_1'$. The same assertion holds for p_0'. Without loss of generality, suppose $p_0 = p_0' = p/d$. Let $g = \gcd(p_0, d)$, $p_0 = g p_1$ and $d = g d_0$. Then $\gcd(p_1, d_0) = 1$, and there exist integers x_0 and k_0 satisfying $x_0 p_1 + k_0 d_0 = 1$. By [Lemma 5.2](), $\gcd(g, a) = \gcd(g, b) = 1$. There exist integers a_0 and k_1 satisfying $a_0 a + k_1 g = b$ and $\gcd(a_0, g) = 1$. Suppose $x = (k_1 - k_2 a) x_0$ and $y = k_2 g + a_0$ for some integer k_2. Then

$$\begin{aligned} x p_0 + y a &\equiv (k_1 - k_2 a) x_0 g p_1 + (k_2 g + a_0) a \\ &\equiv (k_1 - k_2 a)(1 - k_0 d_0) g + (k_2 g + a_0) a \equiv k_1 g + a_0 a \equiv b \pmod{g d_0}. \end{aligned}$$

The map

$$f : \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \quad \text{given by } t \mapsto t + xr \text{ and } r \mapsto yr$$

is an isomorphism if and only if $\gcd(y, d) = 1$. Since $f(t + ar) = t + (xp_0 + ya)r$, this lemma follows from the next proposition. $\qquad\square$

Proposition 5.4 *Suppose integers a_0 and g satisfying $\gcd(a_0, g) = 1$. For any integer d there exists an integer k_2 satisfying $\gcd(y, d) = 1$, where $y = k_2 g + a_0$.*

Proof If q is a prime number satisfying $p \mid \gcd(g, d)$, then a_0 is not divisible by q and neither is y because $\gcd(a_0, g) = 1$. Then $\gcd(y, d) = \gcd(y, d/q)$. Without loss of generality, suppose $\gcd(g, d) = 1$. By the Chinese remainder theorem, the following congruence equations have a solution y:

$$y \equiv a_0 \pmod{g}, \quad y \equiv 1 \pmod{d}.$$

Then $\gcd(y, d) = 1$. We know that $k_2 = (y - a_0)/g$ satisfies the proposition. $\qquad\square$

From now on, let us fix isomorphisms j_1 and j_2 as in Lemma 5.3. Then the homology classes of meridians and their images under j_i for $i = 1, 2$ can be identified, ie $[m_1]$ and $[m_2]$ are regarded as the same element $[m]$ in $\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$. The following is the key lemma in this section, and is based on results in [42].

Lemma 5.5 *Let K_1 and K_2 be two knots in $Y = L(p, q)$ representing the same homology class. Let j_i be the isomorphisms $H_1(E(K_i); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} = H_1$ as in Lemma 5.3. Then $\tau(K_1) - \tau(K_2)$ can be regarded as an element in $\mathbb{Z}[H_1]/\pm H_1$. Moreover, we have*

$$\tau(K_1) - \tau(K_2) = (1 - [m])g \quad \text{for some } g \in \mathbb{Z}[H_1]/\pm H_1.$$

Proof Note that $\tau(K_i)$ is not a priori an element in $\mathbb{Z}[H_1(E(K_i); \mathbb{Z})]/\pm H_1(E(K_i); \mathbb{Z})$ (see [42, Corollary II.4.3]). However, the difference $\tau(K_1) - \tau(K_2)$ is a well-defined element in $\mathbb{Z}[H_1]/\pm H_1$ under the isomorphisms of group rings induced by j_1 and j_2. To resolve the ambiguity of $\pm H_1$, we can choose an Euler structure and a homology orientation on $E(K_i)$ (see [42, Section I.1]). For any compact 3–manifold with torus boundary, Euler structures are in one-to-one correspondence with spin^c structures related to the Alexander grading. For any closed 3–manifold, Euler structures are in one-to-one correspondence with spin^c structures on the manifold. We omit the choice

of the homology orientation that determines the sign of $\tau(K_i)$, and only consider the choice of the Euler structure for simplicity. For an Euler structure e on M, the Turaev torsion $\tau(M)$ has a representative $\tau(M, e)$.

For $i = 1, 2$, let e_i be Euler structures on $E(K_i)$ inducing the same Euler structure e^Y on Y. Adapting notation from [42, Section II.4.5], suppose the integer $K(e_i)$ satisfies

$$c(e_i) = \frac{e_i}{e_i^{-1}} \in t^{K(e_i)} \text{ Tors } H_1.$$

We can also consider $c(e_i)$ as the Chern class of the spin^c structure on $E(K_i)$ corresponding to e_i. Note that t is the generator of the free part of H_1.

From the correspondence between Euler structures and spin^c structures, it is possible to choose e_i so that $K(e_1) = K(e_2)$. In the proof of [42, Lemma II.4.5.1(i)], we have

$$\tau(E(K_i), e_i) \in \frac{-\Sigma_{H_1}}{t - 1} + \mathbb{Z}[H_1],$$

where $\Sigma_H = \Sigma_{h \in \text{Tors } H_1} h$, so $\tau(K_1, e_1) - \tau(K_2, e_2) \in \mathbb{Z}[H_1]$. Also, in [42, Section II.4], for a 3–manifold M with $b_1(M) = 1$, the polynomial part $[\tau](M, e) \in \left(\frac{1}{2}\mathbb{Z}\right)[H_1]$ of $\tau(M, e)$ is defined by

$$(4) \quad [\tau](M, e) = \left(\tau(M, e) + \frac{\Sigma_{H_1(M)}}{t - 1}\right) \times \begin{cases} t^{\frac{1}{2}(K(e)+1)} & \text{if } K(e) \text{ is odd,} \\ t^{\frac{1}{2}K(e)}\left(\frac{1}{2}(t + 1)\right) & \text{if } K(e) \text{ is even.} \end{cases}$$

By [42, Remark II.4.5.2], for any Euler structure e on M, the polynomial part $[\tau](M, e)$ is in the kernel of the map aug$: \mathbb{Z}[H_1] \to \mathbb{Z}$ that sends elements in H_1 to $1 \in \mathbb{Z}$. Thus,

$$\text{aug}(\tau(K_1, e_1) - \tau(K_2, e_2)) = \text{aug}([\tau](K_1, e_1) - [\tau](K_2, e_2)) = 0.$$

By the $m = 1$ case in [42, Theorem X.4.1], since the map $\kappa: \mathbb{Q}[H_1] \to \mathbb{Q}[H_1]$ that sends x to $x - \text{aug}(x)\Sigma_{H_1}/|H_1|$ is trivial, we have

$$\text{pr}(\tau(K_1, e_1) - \tau(K_2, e_2)) = -([K_1] - 1)\tau(Y, e^Y) + ([K_2] - 1)\tau(Y, e^Y) = 0,$$

where pr is the map in the following proposition. Also from the following proposition, there is an element $g \in \mathbb{Z}[H_1]$ such that

$$\tau(K_1, e_1) - \tau(K_2, e_2) = (1 - [m])g.$$

Since $\tau(K_1, e_1) - \tau(K_2, e_2)$ reduces to $\tau(K_1) - \tau(K_2)$ in $\mathbb{Z}[H_1]/\pm H_1$, we obtain the equation for elements in $\mathbb{Z}[H_1]/\pm H_1$. $\qquad \square$

Proposition 5.6 *Let* $\mathrm{pr}\colon \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ *be the map between group rings induced by the composition of maps*

$$\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \xrightarrow{\cong} H_1(E(K_i);\mathbb{Z}) \to H_1(E(K_i);\mathbb{Z})/([m_i]) \xrightarrow{\cong} H_1(Y;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z}.$$

Then the kernel of pr *is the ideal generated by* $1 - [m]$.

Proof Suppose $\mathbb{Z}/p\mathbb{Z} = \{s_1, \dots, s_p\}$ and suppose $H = \sum_{i=1}^{k} a_i h_i$ is an element in the kernel of pr, where $a_i \in \mathbb{Z}$ and $h_i \in \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$. Let $\Theta_H(s_j)$ be the set consisting of all elements h_i satisfying $\mathrm{pr}(h_i) = s_j$ in the summation defining H. Then $\sum_{h_i \in \Theta_H(s_j)} a_i h_i$ is also in the kernel of pr for any j. Without loss of generality, suppose $\mathrm{pr}(h_i) = s_1$ for any h_i in the summation of H. By definition of the map pr, for any i, we have $h_i = [m]^{\alpha(i)} h_1$ for some integer $\alpha(i)$. Then

$$H = \sum_{j=0}^{k'} b_j [m]^j h_1$$

for some integer k'. Since H is in the kernel of pr, we have

$$\sum_{j=0}^{k'} b_j = 0.$$

Thus, the polynomial

$$\sum_{j=0}^{k'} b_j x^j$$

has a root $x = 1$. In other words, $\sum_{j=0}^{k'} b_j x^j = (1-x)g(x)$ for some polynomial $g(x)$. Then we have $H = (1 - [m])g([m])h_1$ and conclude the proposition.

There is another quick proof from the referee. The functor that takes a group to its group ring is left-adjoint to the functor that takes a commutative ring to its group of units. The quotient $\mathbb{Z}/p\mathbb{Z}$ is the colimit of the diagram $\mathbb{Z} \rightrightarrows \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$, where one map is $1 \mapsto [m]$ and the other is the zero map. Then the proposition follows from the fact that left-adjoints preserve colimits.   □

Lemma 5.7 [39, Proposition 2.1] *Suppose* K *is a knot in* $Y = L(p,q)$ *and let* $H_1 = H_1(E(K);\mathbb{Z})$. *Then*

$$\chi(\widehat{HFK}(Y,K)) = (1-[m])\tau(K) \in \mathbb{Z}[H_1]/\pm H_1.$$

Theorem 5.8 Let K_1 and K_2 be two knots representing the same homology class in $Y = L(p,q)$. Suppose $H_1(E(K_i); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} = H_1$ as in Lemma 5.3. After shifting Alexander gradings on $\widehat{HFK}(Y, K_i)$ for $i = 1, 2$, the difference of their Euler characteristics satisfies the following condition: for any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, there exists a Laurent polynomial $f(x) \in \mathbb{Z}[x, x^{-1}]$ and an element $\tilde{s} \in H_1$ such that

$$\chi(\widehat{HFK}(Y, K_1, \mathfrak{s})) - \chi(\widehat{HFK}(Y, K_2, \mathfrak{s})) = ([m] - 1)^2 f([m])\tilde{s}.$$

Proof Note that $\chi(\widehat{HFK}(Y, K_i))$ is an element in $\mathbb{Z}[H_1]$ up to equivalence. Fixing the Alexander grading on $\widehat{HFK}(Y, K_i)$ is equivalent to choosing a representative of $\chi(\widehat{HFK}(Y, K_i))$ in $\mathbb{Z}[H_1]$. By Lemma 5.5 and Lemma 5.7, after shifting Alexander gradings, there exists some $g \in \mathbb{Z}[H_1]/\pm H_1$ such that

$$\chi(\widehat{HFK}(Y, K_1)) - \chi(\widehat{HFK}(Y, K_2)) = (1 - [m])(\tau(K_1) - \tau(K_2)) = ([m] - 1)^2 g.$$

Choose a lift $\tilde{g}$ of g in $\mathbb{Z}[H_1]$. It can be written as the sum $\tilde{g} = \sum_{j=1}^{p} g_j$, where g_j contains terms that are in the preimage of $s_j \in H_1(Y; \mathbb{Z})$ under the map

$$H_1 \to H_1(Y; \mathbb{Z}) = \{s_1, \ldots, s_p\}.$$

For any j, there exists a Laurent polynomial $f_j(x)$ and an element $\tilde{s}_j \in H_1$ such that $g_j = f_j([m])\tilde{s}_j$. Thus, the above equation can be decomposed into spin^c structures, which induces the theorem. $\qquad\square$

Remark 5.9 For constrained knots K_1 and K_2, the group $\widehat{HFK}(Y, K_i)$ can be chosen as the canonical representative in Section 4, meaning we consider the absolute Alexander grading mentioned in the introduction.

Proof of Theorem 1.5 We choose the isomorphisms $H_1(E(K_i); \mathbb{Z}) \cong H_1$ considered in Lemma 5.3. By Lemma 4.2, for a constrained knot $K_i \subset Y$ and a spin^c structure $\mathfrak{s}$ on Y, there is a symmetrized Alexander polynomial $\Delta_i(t)$ of a 2–bridge knot, so that

$$\chi(\widehat{HFK}(Y, K_i, \mathfrak{s})) \sim \Delta_i([m]).$$

Since the Alexander grading reduces to the grading induced by spin^c structures under the map $H_1(E(K_i); \mathbb{Z}) \to H_1(Y; \mathbb{Z})$, we know Alexander gradings of nontrivial summands of $\widehat{HFK}(Y, K_i, \mathfrak{s})$ correspond to the spin^c structure $\mathfrak{s}$. By definition of the equivalence on $\mathbb{Z}[H_1]$, there exists an element $\tilde{s} \in H_1$ in the preimage of $\mathfrak{s}$ such that

$$(5) \qquad\qquad \chi(\widehat{HFK}(Y, K_i, \mathfrak{s})) = \pm\Delta_i([m])[m]^{\gamma_i}\tilde{s},$$

where γ_i is an integer. Write $f_i(x) = \pm\Delta_i(x)x^{\gamma_i}$ for simplicity. Since $\Delta_i(t)$ is symmetrized, the middle grading is the grading of $[m]^{\gamma_i}\tilde{s}$. Note that the multiplication in $\mathbb{Z}[H_1]$ corresponds to the addition in H_1. Then we have

$$A(K_1, \mathfrak{s}) - A(K_2, \mathfrak{s}) = (\gamma_1[m] + \tilde{s}) - (\gamma_2[m] + \tilde{s}) = (\gamma_1 - \gamma_2)[m] \in H_1.$$

By Theorem 5.8, there is a Laurent polynomial $f(x) \in \mathbb{Z}[x, x^{-1}]$ such that

$$f_1(x) - f_2(x) = (x-1)^2 f(x).$$

Hence for a large integer N, there is a polynomial $f_0(x)$ such that

$$x^N(f_1(x) - f_2(x)) = (x-1)^2 f_0(x).$$

Substituting $x = 1$ gives $f_1(1) = f_2(1)$, that is signs in (5) are the same for $i = 1, 2$. Consider derivatives at $x = 1$:

$$0 = \frac{d(x^N(f_1(x) - f_2(x)))}{dx} = N(f_1(1) - f_2(1)) + \frac{df_1}{dx}(1) - \frac{df_2}{dx}(1)$$

$$= \pm\left(\frac{d\Delta_1(x)}{dx}(1) - \frac{d\Delta_2(x)}{dx}(1) + \gamma_1\Delta_1(1) - \gamma_2\Delta_2(1)\right) = \gamma_1 - \gamma_2,$$

where the last equation follows from $\Delta_i(t) = \Delta_i(t^{-1})$ and $\Delta_i(1) = 1$. Thus, we have $A(K_1, \mathfrak{s}) = A(K_2, \mathfrak{s})$. $\qquad\square$

Proof of Theorem 1.6 This follows from the proof of Theorem 1.5 with $\Delta_i(t) = 1$. $\square$

6 Classification

The main result in this section is the proof of the sufficient part of Theorem 1.2. The following lemma enables us to prove it by considering knot groups, ie fundamental groups of knot complements.

Lemma 6.1 [43] *Let M_1 and M_2 be Haken manifolds with torus boundaries. If there is an isomorphism $\psi : \pi_1(M_1) \to \pi_1(M_2)$ that induces an isomorphism*

$$\psi|_{\pi_1(\partial M_1)} : \pi_1(\partial M_1) \to \pi_1(\partial M_2),$$

then there exists a diffeomorphism $\psi_0 : (M_1, \partial M_1) \to (M_2, \partial M_2)$ inducing ψ.

In addition, if M_1 and M_2 are two knot complements and ψ sends the meridian of one knot to the meridian of the other knot, then two knots are equivalent.

A constrained knot is defined by a doubly pointed Heegaard diagram, from which it is easy to obtain a Heegaard diagram of the knot complement similar to the case in Figure 3, right. The Heegaard diagram is related to the handlebody decomposition of the corresponding 3–manifold, and then also related to the cell complex of the corresponding 3–manifold. Thus, it is possible to obtain a presentation of the fundamental group from the Heegaard diagram. We show this presentation explicitly:

Suppose $K = C(p, q, l, u, v)$ is a constrained knot with $u > 2v \geq 0$. Suppose $(T^2, \alpha_1, \beta_1, z, w)$ is the standard diagram of K. Let Σ be the surface of genus two obtained by attaching a 1–handle at basepoints z and w. Suppose α_2 is the curve on Σ that is a union of an arc connecting z to w in $T^2 - \alpha_1$ and an arc on the attached handle; see Figure 6. Suppose $\beta = \beta_1$. Then $(\Sigma, \{\alpha_1, \alpha_2\}, \beta)$ is a Heegaard diagram of $E(K)$.

Let the innermost rainbow R_0 around w be oriented from the right boundary point x_r to the left boundary point x_l. This induces an orientation of β. Let α_1 and α_2 be oriented from the left vertical edge to the right vertical edge in the new diagram C of the constrained knot.

Suppose s and t correspond to cores of α_1–handle and α_2–handle, respectively. In the above orientation, we can obtain a presentation $\pi_1(E(K)) \cong \langle s, t \mid \omega = 1 \rangle$, where the word ω is given in the following way:

  (i)   Starting at x_l and traveling along β, suppose intersection points of $\beta \cap (\alpha_1 \cup \alpha_2)$ are ordered as $x_1, x_2, \ldots, x_m$.

 (ii)   If x_i is an intersection point of α_1 and β it corresponds to a word $s^{\pm 1}$, where the sign depends on the contribution of x_i in the algebraic intersection number $\alpha_1 \cap \beta$.

(iii)   If x_i is an intersection point of α_2 and β it corresponds to a word $t^{\pm 1}$, where the sign depends on the contribution of x_i in the algebraic intersection number $\alpha_2 \cap \beta$.

 (iv)   The word ω is obtained from $x_1 x_2 \cdots x_m$ by replacing x_i by corresponding words in $\{s, s^{-1}, t, t^{-1}\}$.

The word $\omega(p, q, l, u, v) = \omega(C(p, q, l, u, v))$ in the above setting is called the *standard relation* of a constrained knot $C(p, q, l, u, v)$. We begin by understanding the standard relation of a 2–bridge knot. For fixed integers (u, v), let $\epsilon_i = (-1)^{\lfloor iv/u \rfloor}$.

Lemma 6.2 *For the constrained knot $C(1, 0, 1, u, v) \cong \mathfrak{b}(u, v)$, the standard relation ω is $s^{\epsilon_1} t^{\epsilon_2} s^{\epsilon_3} \cdots s^{\epsilon_{2u-1}} t^{\epsilon_{2u}}$.*

Proof This is from the relation between the Schubert normal form and the Heegaard diagram of the 2–bridge knot. Note that the formula of the Alexander polynomial in Proposition 2.9 follows from this presentation and Fox calculus [42, Chapter II].   □

For fixed integers (p, q, l) with $q \in [1, p - 1]$, $l \in [1, p]$ and $\gcd(p, q) = 1$, suppose the integer $k \in (0, p]$ satisfies $k - 1 \equiv (l - 1)q \pmod{p}$ and the integer $q_i \in [0, p)$ satisfies $q_i \equiv iq \pmod{p}$. Define

$$\theta_i = \theta_i(p, q, l) = \begin{cases} 1 & \text{if } q_i \in [0, k), \\ 0 & \text{if } q_i \in [k, p), \end{cases}$$

$$s_*(p, q, l) = st^{\theta_l} st^{\theta_{l+1}} s \cdots st^{\theta_{p-1}} s,$$

$$t_*(p, q, l) = t^{\theta_0} st^{\theta_1} s \cdots st^{\theta_{l-1}}.$$

In particular, we have $\theta_0 = 1$ and $\theta_{l-1} = 1$. Note that the integer q in $s_*(p, q, l)$ or $t_*(p, q, l)$ does not correspond to the parameter q in $C(p, q, l, u, v)$. Indeed, the constrained knot $C(p, q, l, u, v)$ corresponds to $s_*(p, q', l)$ and $t_*(p, q', l)$, where $qq' \equiv 1 \pmod{p}$. We can see this fact from the following proposition.

Proposition 6.3 *For $K = C(p, q, l, u, v)$, suppose that the integer $q' \in [0, p)$ satisfies $qq' \equiv 1 \pmod{p}$. Suppose $s_* = s_*(p, q', l)$ and $t_* = t_*(p, q', l)$. Define*

$$t_{\#}^{\epsilon_i} = \begin{cases} t^{\epsilon_i} & \text{if } \epsilon_{i-1} = -\epsilon_{i+1}, \\ t_*^{\epsilon_i} & \text{if } \epsilon_{i-1} = \epsilon_{i+1}. \end{cases}$$

Then the standard relation of K is $\omega(p, q, l, u, v) = s_*^{\epsilon_1} t_{\#}^{\epsilon_2} s_*^{\epsilon_3} \cdots s_*^{\epsilon_{2u-1}} t_{\#}^{\epsilon_{2u}}$.

Proof The standard diagram of $C(p, q, l, u, v)$ generalizes the standard diagram of $C(1, 0, 1, u, v)$. Then $\omega(p, q, l, u, v)$ can be obtained from $\omega(1, 0, 1, u, v)$ by replacing s and t by some words. We figure out the replacement as follows.

Suppose that the integer $k \in (0, p]$ satisfies $(k - 1)q \equiv l - 1 \pmod{p}$, which coincides with the definition of k for (p, q', l) before this proposition. Note that we define q_i' by $qq_i' \equiv i \pmod{p}$ since we consider (p, q', l) rather than (p, q, l).

Consider the new diagram C of K mentioned in Section 3; see Figure 4. There are regions D_j for $j \in \mathbb{Z}/p\mathbb{Z}$, where the right edge of D_j is glued to the left edge of D_{j+q}. Consider the part of α_2 on T^2 that connects z to w. It goes across regions in the order

$$D_1, D_{q+1}, D_{2q+1}, \ldots, D_l.$$

By definition of k, there are k regions in the above sequence. By definition of q'_i, any region D_j in the above sequence lies at the $(q'_{j-1}+1)^{\text{th}}$ position, so $q'_j < k$. For example, $q'_0 = 0$ implies that D_1 lies at the first position and $q'_{l-1} = k-1$ implies that D_l lies at the kth position. Then $\theta_j = 1$ if and only if $\alpha_2 \cap D_{j+1}$ is nonempty.

Then the word $s_*(p, q', l)$ corresponds to intersection points of $\beta \cap (\alpha_1 \cup \alpha_2)$ on an arc component of $\beta \cap \left(\bigcup_{j=l+1}^{p-1} D_j\right)$. The word $t_*(p, q', l)$ corresponds to intersection points of $\beta \cap (\alpha_1 \cup \alpha_2)$ on an arc component of $\beta \cap \left(\bigcup_{j=1}^{l} D_j\right)$ that is also a subarc of a stripe.

Thus, we can replace s by $s_* = s_*(p, q', l)$. When $\epsilon_{i-1} = -\epsilon_{i+1}$, the corresponding intersection point related to t^{ϵ_i} is on the rainbow, so we just replace t^{ϵ_i} by t^{ϵ_i} itself. When $\epsilon_{i-1} = \epsilon_{i+1}$, the corresponding intersection point related to t^{ϵ_i} is on the stripe, so we replace t^{ϵ_i} by $t_*^{\epsilon_i} = t_*^{\epsilon_i}(p, q', l)$. This is how $t_\#^{\epsilon_i}$ is defined. $\qquad\square$

Suppose $K_1 = C(p, q, l, u, v)$ and $K_2 = C(p, q', l, u, v)$, where $qq' \equiv 1 \pmod{p}$ and $l \in \{2, p\}$. Proposition 6.3 provides presentations of $\pi_1(E(K_1))$ and $\pi_1(E(K_2))$. We will construct an explicit isomorphism $\pi_1(E(K_1)) \cong \pi_1(E(K_2))$ based on the standard relations. First of all, let us introduce some notation:

Given words w_1 and w_2 made by s and t, let $h_{w_1, w_2} = h(w_1, w_2)$ be a map on words such that for any word ω made by s and t, the word $h_{w_1, w_2}(\omega)$ is obtained from ω by replacing s and t by w_1 and w_2, respectively. For any integer n, define maps

$$f_1^n = h(s, s^n t), \quad f_2^n = h(t^n s, t), \quad g_1^n = h(s, ts^n), \quad g_2^n = h(s^n t, t)$$

and

$$h_0 = h(t, s), \quad h_1 = h(t, s^{-1}), \quad h_2 = h_1 \circ h_1 = h(s^{-1}, t^{-1}).$$

The map f_1^n induces an isomorphism $\langle s, t \mid \omega \rangle \cong \langle s, t \mid f_1^n(\omega) \rangle$ by mapping t to $s^n t$ and s to s, which is still denoted by f_1^n. A similar argument applies to f_2^n. For m odd, let $f_m^n = f_1^n$. For m even, let $f_m^n = f_2^n$. Given integers $p, q > 0$, suppose

$$\frac{q}{p} = [a_0; a_1, a_2, \ldots, a_m] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

is the unique continued fraction of q/p with $a_i > 0$ and $a_m > 1$. Define

$$f^{q/p} = f_m^{-a_m+1} \circ f_{m-1}^{-a_{m-1}} \circ \cdots \circ f_1^{-a_1} \circ f_0^{-a_0} \quad \text{and} \quad F^{q/p} = f_1^1 \circ f_2^{-1} \circ f^{q/p}.$$

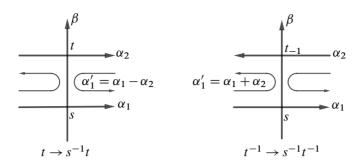The maps g_m^n, $g^{q/p}$ and $G^{q/p}$ are defined similarly based on g_1^n and g_2^n.

Figure 8: Examples of handle slides.

Remark 6.4 The isomorphisms f_m^n and g_m^n can be achieved by handle slides of α curves in the Heegaard diagram of the knot complement. Indeed, if there are two consecutive intersection points x_i and x_{i+1} in the definition of the standard relation that correspond to s and t, respectively, then the arc of β between x_i and x_{i+1} can be used for the handle slide. If α_1 is slid over α_2, then the relation ω becomes $f_1^{-1}(\omega)$. If α_2 is slid over α_1, then the relation ω becomes $g_2^{-1}(\omega)$. Moreover, when

$$(x_i, x_{i+1}) \to (s,t),\ (s,t^{-1}),\ (s^{-1},t),\ (s^{-1},t^{-1}),\ (t,s),\ (t,s^{-1}),\ (t^{-1},s),\ (t^{-1},s^{-1}),$$

where $\to$ implies the replacement considered in the definition of the standard relation, then the corresponding maps are

$$\begin{aligned}
(f_1^{-1}, g_2^{-1}), \quad (g_1^1, g_2^1), \quad (f_1^1, f_2^1), \quad (g_1^{-1}, f_2^{-1}), \\
(f_2^{-1}, g_1^{-1}), \quad (g_2^1, g_1^1), \quad (f_2^1, f_1^1), \quad (g_2^{-1}, f_1^{-1}),
\end{aligned}$$

respectively. Two examples are shown in Figure 8.

The proof of the following lemma follows directly from definitions of maps.

Lemma 6.5 *There are relations between maps*

   (i) $h_0 \circ h_0 = h_2 \circ h_2 = \mathrm{id}$,

   (ii) $f_1^n \circ h_1 = h_1 \circ f_2^{-n}$ *and* $f_2^n \circ h_1 = h_1 \circ g_1^{-n}$,

   (iii) $g_1^n \circ h_1 = h_1 \circ g_2^{-n}$ *and* $g_2^n \circ h_1 = h_1 \circ f_1^{-n}$.

In the following lemmas, integers p, q and q' satisfy

$$p > 0, \quad q, q' \in [1, p-1], \quad \gcd(p,q) = 1 \quad \text{and} \quad qq' \equiv 1 \ (\mathrm{mod}\ p).$$

Lemma 6.6 *The following equations hold:*

(6) $$f^{q/p}(s_*(p,q,2)ts) = ts \quad \text{and} \quad f^{q/p}(s_*(p,q,2)st) = st,$$

(7) $$g^{q/p}(tss_*(p,q,2)) = ts \quad \text{and} \quad g^{q/p}(sts_*(p,q,2)) = st.$$

Proof If $l = 2$, by definition $s_*(p,q,2) = st^{\theta_2}st^{\theta_3}s \cdots st^{\theta_{p-1}}s$, where $\theta_i = \theta_i(p,q,2)$. Suppose that the integer k satisfies $k - 1 \equiv (l-1)q \pmod{p}$. We know that $k = q + 1$. Suppose

$$\frac{q}{p} = [0; a_1, a_2, \ldots, a_m]$$

with $a_i > 0$ and $a_m > 1$. We prove (6) by induction on m.

If $m = 1$, then $q = 1$ and $p = a_1$. Thus $s_*(p,q,2) = s^{a_1 - 1}$ and $f^{q/p} = f_1^{-(a_1-1)}$ by definition. It can be checked directly that (6) holds.

Suppose (6) holds for $m = m_0 - 1$. Consider integers q_i satisfying $q_i \equiv iq \pmod{p}$. Since $\gcd(p,q) = 1$, if $q_i \equiv iq \equiv q \pmod{p}$, then $i = 1$. So $q_i \neq q$ for $i \in [2, p-1]$. Since $k = q + 1$, the condition $q_i \in [0, k)$ is the same as $q_i \in [0, q)$ for $i \in [2, p-1]$. Thus $\theta_i(p,q,2) = 1$ if and only if

$$\left\lfloor \frac{iq}{p} \right\rfloor - \left\lfloor \frac{(i-1)q}{p} \right\rfloor = 1.$$

In other words, we have

$$\theta_i(p,q,2) = \left\lfloor \frac{iq}{p} \right\rfloor - \left\lfloor \frac{(i-1)q}{p} \right\rfloor \text{ for } i \in [2, p-1].$$

If $\theta_i(p,q,2) = 1$, there is some integer $j \in [1, q-1]$ such that

$$i = \left\lfloor \frac{jp}{q} \right\rfloor + 1 = ja_1 + \left\lfloor \frac{jr}{q} \right\rfloor + 1,$$

where

$$\frac{r}{q} = [0; a_2, a_3, \ldots, a_{m_0}].$$

Let $j_1 = j$ and $j_2 = j - 1$ for $j \in [2, q-1]$. Then we have

$$\left(j_1 a_1 + \left\lfloor \frac{j_1 r}{q} \right\rfloor + 1 \right) - \left(j_2 a_1 + \left\lfloor \frac{j_2 r}{q} \right\rfloor + 1 \right) = a_1 + \left\lfloor \frac{j_1 r}{q} \right\rfloor - \left\lfloor \frac{j_2 r}{q} \right\rfloor = a_1 + \theta_j(q,r,2).$$

Thus

$$s_*(p,q,2)ts = s^{a_1}ts^{a_1}s^{\theta_2(q,r,2)}ts^{a_1}s^{\theta_3(q,r,2)}t \cdots s^{a_1}s^{\theta_{q-2}(q,r,2)}ts^{a_1}s^{\theta_{q-1}(q,r,2)}ts^{a_1}ts$$

$$= (s^{a_1}t)s^{\theta_2(q,r,2)}(s^{a_1}t)s^{\theta_3(q,r,2)}(s^{a_1}t)\cdots s^{\theta_{q-2}(q,r,2)}(s^{a_1}t)s^{\theta_{q-1}(q,r,2)}$$
$$\cdot (s^{a_1}t)(s^{a_1}t)s$$

$$= h_{s^{a_1}t,s}(s_*(q,r,2)st) = f_1^{a_1} \circ h_0(s_*(q,r,2)st),$$

where the second equality follows from the fact that $\theta_i(p, q, 2) = 0$ if $i < a_1$. Similarly,

$$s_*(p, q, 2)st = s^{a_1}ts^{a_1}s^{\theta_2(q,r,2)}ts^{a_1}s^{\theta_3(q,r,2)}t\cdots s^{a_1}s^{\theta_{q-2}(q,r,2)}ts^{a_1}s^{\theta_{q-1}(q,r,2)}ts^{a_1}st$$

$$= (s^{a_1}t)s^{\theta_2(q,r,2)}(s^{a_1}t)s^{\theta_3(q,r,2)}(s^{a_1}t)\cdots s^{\theta_{q-2}(q,r,2)}(s^{a_1}t)s^{\theta_{q-1}(q,r,2)}$$
$$\cdot(s^{a_1}t)s(s^{a_1}t)$$

$$= h_{s^{a_1}t,s}(s_*(q, r, 2)ts) = f_1^{a_1} \circ h_0(s_*(q, r, 2)ts).$$

By the inductive assumption, we have

$$f^{r/q}((s_*(p, q, 2)ts) = ts \quad \text{and} \quad f^{r/q}((s_*(p, q, 2)st) = st.$$

Since $f^{q/p} = h_0 \circ f^{r/q} \circ h_0 \circ f_1^{-a_1}$ and $h_0 \circ h_0 = \text{id}$, we have

$$f^{q/p}((s_*(p, q, 2)ts) = h_0 \circ f^{r/q}(s_*(q, r, 2)st) = h_0(st) = ts,$$
$$f^{q/p}((s_*(p, q, 2)st) = h_0 \circ f^{r/q}(s_*(q, r, 2)ts) = h_0(ts) = st.$$

By a similar method, it can be proven that

$$tss_*(p, q, 2) = g_1^{a_1} \circ h_0(sts_*(q, r, 2)) \quad \text{and} \quad sts_*(p, q, 2) = g_1^{a_1} \circ h_0(tss_*(q, r, 2)).$$

Then by induction, (7) holds. $\square$

Lemma 6.7 *The following equations hold*:

$$F^{q/p}(t) = f_1^1 \circ f_2^{-1} \circ f^{q/p}(t) = h_0(s_*(p, q', 2)ts),$$
$$G^{q/p}(t) = g_1^1 \circ g_2^{-1} \circ g^{q/p}(t) = h_0(sts_*(p, q', 2)).$$

Proof The proofs of the two equations are similar. We only show the proof of the first equation. By the proof of Lemma 6.6, we know

$$\theta_i(p, q, 2) = \left\lfloor \frac{iq}{p} \right\rfloor - \left\lfloor \frac{(i-1)q}{p} \right\rfloor \quad \text{for } i \in [2, p-1].$$

Thus $\theta_i(p, q, 2) = 0$ if and only if

$$\left\lfloor \frac{i(q-p)}{p} \right\rfloor - \left\lfloor \frac{(i-1)(q-p)}{p} \right\rfloor = \left\lfloor \frac{iq}{p} \right\rfloor - \left\lfloor \frac{(i-1)q}{p} \right\rfloor - 1 = -1.$$

This is equivalent to

$$\left\lfloor \frac{i(p-q)}{p} \right\rfloor - \left\lfloor \frac{(i-1)(p-q)}{p} \right\rfloor = 1,$$

ie $\theta_i(p, p-q, 2) = 1$. Then

$$f_1^{-1} \circ h_0(s_*(p, q, 2)ts) = s^{-1}ts^{-\theta_2(p,p-q,2)}t \cdots ts^{-\theta_{p-1}(p,p-q,2)}tt$$
$$= s^{-1}h_1(s_*(p, p-q)st)s = t^{-1}h_1(sts_*(p, p-q))t.$$

Suppose $q/p = [0; a_1, a_2, \ldots, a_m]$ with $a_i > 0$ and $a_m > 1$. We have

$$f_2^{-1} \circ f^{q/p} = \begin{cases} f_2^{-1} \circ f_1^{-a_m+1} \circ f_2^{-a_{m-1}} \circ \cdots \circ f_2^{-a_2} \circ f_1^{-a_1} & \text{if } m \text{ is odd,} \\ f_2^{-a_m} \circ f_1^{-a_{m-1}} \circ \cdots \circ f_2^{-a_2} \circ f_1^{-a_1} & \text{if } m \text{ is even.} \end{cases}$$

By the extended Euclidean algorithm,

$$\frac{p - q'}{p} = \begin{cases} [0; 1, a_m - 1, a_{m-1}, \ldots, a_2, a_1] & \text{if } m \text{ is odd,} \\ [0; a_m, a_{m-1}, \ldots, a_2, a_1] & \text{if } m \text{ is even.} \end{cases}$$

It can be proven by induction on n that for $b/a = [0; b_1, b_2, \ldots, b_{2n-1}, b_{2n}]$,

(8) $$f_2^{-b_1} \circ f_1^{-b_2} \circ \cdots \circ f_2^{-b_{2n-1}} \circ f_1^{-b_{2n}}(t) = h_1(ts_*(a,b)s).$$

Indeed, if $n = 1$, then $f_2^{-b_2} \circ f_1^{-b_1}(t) = (t^{-b_2}s)^{-b_1}t = (s^{-1}t^{b_2})^{b_1}t$. Equation (8) is clear.

Suppose (8) holds for $n = n_0 - 1$. Let

$$\frac{b'}{a'} = [0; b_2, \ldots, b_{2n_0-1}, b_{2n_0}] \quad \text{and} \quad \frac{b''}{a''} = [0; b_3, \ldots, b_{2n_0-1}, b_{2n_0}].$$

By the proof of Lemma 6.6,

$$\begin{aligned} tf_1^{b_1}(s_*(a'', b'', 2)st)t^{-1} &= th_0(s_*(a', b', 2)ts)t^{-1} \\ &= s^{-1}h_0(tss_*(a', b', 2))s = s^{-1}g_1^{b_1}(sts_*(a'', b'', 2))s. \end{aligned}$$

Thus

$$\begin{aligned} f_2^{-b_1} \circ f_1^{-b_2} \circ h_1(ts_*(a, b, 2)stt^{-1}) \\ &= f_2^{-b_1} \circ h_1 \circ f_2^{b_2}(tf_2^{-b_2} \circ f_1^{-b_1}(s_*(a'', b'', 2)st)t^{-1}) \\ &= f_2^{-b_1} \circ h_1(tf_1^{-b_1}(s_*(a'', b'', 2)st)t^{-1}) \\ &= h_1 \circ g_1^{b_1} \circ (s^{-1}g_1^{b_1}(sts_*(a'', b'', 2))s) \\ &= h_1(s^{-1}(sts_*(a'', b'', 2))s) = h_1(ts_*(a'', b'', 2)s). \quad \square \end{aligned}$$

Remark 6.8 By Remark 6.4, the map $f^{q/p}$ can be regarded as a sequence of handle slides. Consider the matrix of algebraic intersection points

$$\begin{bmatrix} [\alpha_1] \cdot p[a] & [\alpha_2] \cdot p[a] \\ [\alpha_1] \cdot [m] & [\alpha_2] \cdot [m] \end{bmatrix},$$

where a and m are curves in Figure 6. The maps f_1^n and f_2^n induce column transformations of this matrix, which are still denoted by f_1^n and f_2^n. Then

$$f^{q/p}\left(\begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ q' - p & q' \end{bmatrix} \quad \text{and} \quad F^{q/p}\left(\begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ q' & p \end{bmatrix}.$$

Indeed, the definitions of $f^{q/p}$ and $F^{q/p}$ come from the extend Euclidean algorithm for calculating $\gcd(p, q)$ (see the proof of Lemma 6.7).

Proposition 6.9 *Up to circular permutation,*

$$h_0 \circ F^{q/p}(\omega(p, q', 2, u, v)) = \begin{cases} h_2(\omega(p, q, 2, u, v)) & \text{if } v \text{ is odd,} \\ \omega(p, q, 2, u, v) & \text{if } v \text{ is even.} \end{cases}$$

Proof Suppose $a = s_*(p, q, 2)$ and $b = s_*(p, q', 2)$. Then

$$t_* = t_*(p, q, 2) = t_*(p, q', 2) = tst$$

and

$$\omega(p, q', 2, u, v) = a^{\epsilon_1} t_{\#}^{\epsilon_2} a^{\epsilon_3} \cdots t_{\#}^{\epsilon_{2u}}, \quad \omega(p, q, 2, u, v) = b^{\epsilon_1} t_{\#}^{\epsilon_2} b^{\epsilon_3} \cdots t_{\#}^{\epsilon_{2u}}.$$

The word $a^{\epsilon_{i-1}} t_{\#}^{\epsilon_i} a^{\epsilon_{i+1}}$ is one of

(i) $atsta = (ats)ta$ and $a^{-1}(tst)^{-1}a^{-1} = a^{-1}t^{-1}(ats)^{-1}$,

(ii) $ata^{-1} = (ats)t(ast)^{-1}$ and $at^{-1}a^{-1} = (ast)t^{-1}(ats)^{-1}$,

(iii) $a^{-1}ta$ and $a^{-1}t^{-1}a$.

Thus $\omega(p, q', 2, u, v) = a_{\#}^{\epsilon_1} t^{\epsilon_2} a_{\#}^{\epsilon_3} \cdots t^{\epsilon_{2u}}$, where

$$a_{\#}^{\epsilon_i} = \begin{cases} (ats)^{\epsilon_i} & \text{if } \epsilon_i = \epsilon_{\epsilon_i+i}, \\ (ast)^{\epsilon_i} & \text{if } \epsilon_i = -\epsilon_{\epsilon_i+i}. \end{cases}$$

By Lemma 6.6 and Lemma 6.7,

$$F^{q/p}(ats) = s = h_0(t), \quad F^{q/p}(ast) = t^{-1}st = h_0(s^{-1}ts) \quad \text{and} \quad F^{q/p}(t) = h_0(bts).$$

Thus $h_0 \circ F^{q/p}(\omega(p, q', 2, u, v)) = c_{\#}^{\epsilon_1}(bts)^{\epsilon_2} c_{\#}^{\epsilon_3} \cdots (bts)^{\epsilon_{2u}}$, where

$$c_{\#}^{\epsilon_i} = \begin{cases} t^{\epsilon_i} & \text{if } \epsilon_i = \epsilon_{\epsilon_i+i}, \\ (s^{-1}ts)^{\epsilon_i} & \text{if } \epsilon_i = -\epsilon_{\epsilon_i+i}. \end{cases}$$

The word $(bts)^{\epsilon_{i-1}} c_{\#}^{\epsilon_i}(bts)^{\epsilon_{i+1}}$ is one of

(i) $(bts)t(bts) = b(tst)bts$ and $(bts)^{-1}(t)^{-1}(bts)^{-1} = (bts)^{-1}(tst)^{-1}b^{-1}$,

(ii) $(bts)(s^{-1}ts)(bts)^{-1} = btb^{-1}$ and $(bts)(s^{-1}ts)^{-1}(bts)^{-1} = bt^{-1}b^{-1}$,

(iii) $(bts)^{-1}t(bts)$ and $(bts)^{-1}t^{-1}(bts)$.

Thus

$$h_0 \circ F^{q/p}(\omega(p, q', 2, u, v)) = t_{\#}^{\epsilon_1} b^{\epsilon_2} t_{\#}^{\epsilon_3} \cdots b^{\epsilon_{2u}} = b_{\#}^{\epsilon_{u+1}} t^{\epsilon_{u+2}} b_{\#}^{\epsilon_{u+3}} \cdots b^{\epsilon_{3u}},$$

where the last equality holds up to circular permutation. The proposition follows from the fact that $\epsilon_{u+i} = (-1)^v \epsilon_i$. $\qquad \square$

Proposition 6.10 *Up to circular permutation,*

$$h_0 \circ G^{(p-q)/p}(\omega(p, q', p, u, v)) = \begin{cases} h_2(\omega(p, q, p, u, v)) & \text{if } v \text{ is odd,} \\ \omega(p, q, p, u, v) & \text{if } v \text{ is even.} \end{cases}$$

Proof The essential idea of the proof is the same as that of Proposition 6.9. Now

$$s_*(p, q, p) = s \quad \text{and} \quad t_*(p, q, p) = ts_*(p, p-q, 2)t.$$

Suppose $a = s_*(p, p-q, 2)$ and $b = s_*(p, p-q', 2)$. By analyzing cases of $s^{\epsilon_i-1} t_{\#}^{\epsilon_i} s^{\epsilon_i}$ we get $\omega(p, q', p, u, v) = a_{\#}^{\epsilon_1} t^{\epsilon_2} a_{\#}^{\epsilon_3} \cdots t^{\epsilon_{2u}}$, where

$$a_{\#}^{\epsilon_i} = \begin{cases} (sta)^{\epsilon_i} & \text{if } \epsilon_i = \epsilon_{\epsilon_i+i}, \\ (sat)^{\epsilon_i} & \text{if } \epsilon_i = -\epsilon_{\epsilon_i+i}. \end{cases}$$

Note that $\epsilon_i \in \{\pm 1\}$ and $\epsilon_{\epsilon_i+i} = \epsilon_{\pm+i}$ in the definition of $a_{\#}^{\epsilon_i}$. By Lemma 6.6 and Lemma 6.7, we have $G^{(p-q)/p}(t) = stb$. Thus

$$h_0 \circ G^{(p-q)/p}(\omega(p, q', p, u, v)) = c_{\#}^{\epsilon_1}(stb)^{\epsilon_2} c_{\#}^{\epsilon_3} \cdots (stb)^{\epsilon_{2u}},$$

where

$$c_{\#}^{\epsilon_i} = \begin{cases} t^{\epsilon_i} & \text{if } \epsilon_i = \epsilon_{\epsilon_i+i}, \\ ((stb)^{-1}(sts^{-1})(stb))^{\epsilon_i} & \text{if } \epsilon_i = -\epsilon_{\epsilon_i+i}. \end{cases}$$

By analyzing cases of $(stb)^{\epsilon_i-1} c_{\#}^{\epsilon_i}(stb)^{\epsilon_i-1}(stb)^{\epsilon_i+1}$ we get

$$h_0 \circ G^{(p-q)/p}(\omega(p, q', p, u, v)) = t_{\#}^{\epsilon_1} b^{\epsilon_2} t_{\#}^{\epsilon_3} \cdots b^{\epsilon_{2u}}.$$

Then this proposition follows from a similar argument as in Proposition 6.9. □

Proof of the sufficient part of Theorem 1.2 For $i = 1, 2$, let $K_i = C(p_i, q_i, l_i, u_i, v_i)$, $M_i = E(K_i)$ and suppose (μ_i, λ_i) is the regular basis of ∂M_i. Suppose

$$(p_1, u_1, v_1) = (p_2, u_2, v_2) = (p, u, v), q_1 q_2 \equiv 1 \pmod{p} \quad \text{and} \quad l_1 = l_2 \in \{2, p\}.$$

By knot Floer homology, constrained knots K_i are not unknots in lens spaces. By Proposition 2.1, we know that the M_i are Haken manifolds.

Let $q' = q_1$ and $q = q_2$ in Propositions 6.9 and 6.10. Let ψ be the map from $\pi_1(M_1)$ to $\pi_1(M_2)$ induced by

$$h_0 \circ F^{q/p} \quad \text{if} \ \ l_1 = l_2 = 2,$$
$$h_0 \circ G^{(p-q)/p} \quad \text{if} \ \ l_1 = l_2 = p.$$

By Propositions 6.9 and 6.10, the map ψ is an isomorphism. The meridians μ_i and longitudes λ_i can be isotoped to lie on Heegaard diagrams of M_i so that $\mu_1 = m$ and $\mu_2 = pa$, where a and m are curves in Figure 6. Moreover, suppose that meridians and longitudes are disjoint from β_1. By Remarks 6.4 and 6.8, the map

ψ can be achieved by handle slides of α curves. After handle slides, the meridian and the longitude are still disjoint from β_1, which implies ψ induces an isomorphism $\psi|_{\pi_1(\partial M_1)} \colon \pi_1(\partial M_1) \to \pi_1(\partial M_2)$.

Moreover, for the case $l_1 = l_2 = 2$, note that t corresponds to $\mu_1 \cap (\alpha_1 \cup \alpha_2)$ and $s_*(p, q', 2)ts$ corresponds to $\mu_2 \cap (\alpha_1 \cup \alpha_2)$ in the presentations of the fundamental groups. By Lemma 6.7,

$$\psi|_{\pi_1(\partial M_1)}(\mu_1) = \psi(t) = s_*(p, q', 2)ts = \mu_2.$$

Thus, by Lemma 6.1, we know K_1 is equivalent to K_2.

For the case $l_1 = l_2 = p$, based on Lemma 6.7, the proof is similar. □

7 Magic links

A constrained knot is defined by a doubly pointed Heegaard diagram $(T^2, \alpha_1, \beta_1, z, w)$, where β_1 looks similar to the β curve in the diagram of a 2–bridge knot (see Lemma 4.2 and Proposition 3.5). In this section we provide Dehn surgery descriptions for some families of constrained knots, which is inspired by the relation between constrained knots and 2–bridge knots. The main objects in this section are magic links.

Definition 7.1 Suppose integers u and v satisfy $0 \le v < u$ and $\gcd(u, v) = 1$, and u is odd. Especially, $(u, v) = (1, 0)$ is allowed. A *magic link* $\mathfrak{L}(u, v) = K_0 \cup K_1 \cup K_2$ is a 3–component link linked as shown in Figure 2, left, where K_0 is the 2–bridge knot $\mathfrak{b}(u, v)$ in the standard presentation, and K_1 and K_2 are unknots. For $-u < v < 0$, let $\mathfrak{L}(u, v)$ be the mirror link of $\mathfrak{L}(u, -v)$. Let $\mathfrak{L}(1, 1)$ be the mirror link of $\mathfrak{L}(1, 0)$.

Remark 7.2 The name of magic links is from the fact that the link complement $S^3 - \mathfrak{L}(3, 1)$ is diffeomorphic to the magic manifold studied in [24].

For $i = 1, 2$, suppose integers p_i and q_i satisfy $p_i > 0$ and $\gcd(p_i, q_i) = 1$. Let $M(u, v, p_1/q_1, p_2/q_2)$ and $K_0(u, v, p_1/q_1, p_2/q_2)$ denote the manifold and the resulting knot K_0' obtained by p_i/q_i Dehn surgery on K_i.

Proposition 7.3 *The manifolds $M(u, v, p_1/q_1, p_2/q_2)$ and $M(u, v, p_2/q_2, p_1/q_1)$ are diffeomorphic. Moreover, the knots K_0' in these manifolds are equivalent.*

Proof The components K_1 and K_2 in the magic link switch their positions under the rotation around a vertical line, while K_0 remains unchanged. □
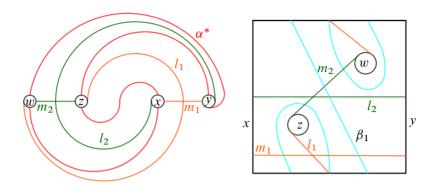
Figure 9: Heegaard diagrams of $E(\mathfrak{L}(3,1))$, where β_1 is omitted in the left figure and α^* is omitted in the right figure.

Remark 7.4 Manifolds $M(u, v, p_1/q_1, p_2/q_2)$ and $M(u, v, p_2/q_2, p_1/q_1)$ will not be distinguished in the rest of the paper. Neither will the corresponding knots K_0 of these manifolds.

Proposition 7.5 *For integers u and v satisfying $0 < v < u$ and $\gcd(u, v) = 1$, and where u is odd, the link $\mathfrak{L}(u, u - v)$ is the mirror link of $\mathfrak{L}(u, v)$. Thus*

$$\mathfrak{L}(u, u - v) \cong \mathfrak{L}(u, -v)$$

and $K_0(u, v, p_1/q_1, p_2/q_2)$ is the mirror image of $K_0(u, u-v, p_1/(-q_1), p_2/(-q_2))$.

Proof Suppose $\mathfrak{b}(u, v)$ is in the standard presentation for

$$\frac{v}{u} = [0; a_1, a_2, \ldots, a_m].$$

Since $(u - v)/u = 1 - v/u$, by adding one positive half-twist on the two left strands, the standard presentation for $[0; -a_1, -a_2, \ldots, -a_m]$ becomes a standard presentation of $\mathfrak{b}(u, u - v)$. After isotoping the link outside twists related to a_i, the link $\mathfrak{L}(u, u - v)$ becomes the mirror link of $\mathfrak{L}(u, v)$. $\square$

Lemma 7.6 *In Figure 9, $(\Sigma_2, \alpha^*, \beta_1)$ are Heegaard diagrams of $E(\mathfrak{L}(3, 1))$. For $i = 1, 2$, the meridian m_i and the longitude l_i of K_i can be isotoped to lie on Σ_2 as in the diagrams. For general integers u and v satisfying $0 < v < u$ and $\gcd(u, v) = 1$, and where u and v are odd, the similar assertion holds when β_1 is replaced by β in the doubly pointed Heegaard diagram of $\mathfrak{b}(u, v)$.*

Figure 10: Meridians and longitudes on the Heegaard surface.

Proof Consider $(u, v) = (3, 1)$. The curve α^* is separating and β_1 is nonseparating. Therefore, the manifold obtained from $\Sigma_2 \times [-1, 1]$ by attaching 2–handles along $\alpha^* \times \{-1\}$ and $\beta_1 \times \{1\}$ has three boundary components, each of which is a torus. Moreover, if two more 2–handles are attached along $m_1 \times \{-1\}$ and $m_2 \times \{-1\}$, the resulting manifold is $E(\mathfrak{b}(3, 1))$. The longitude l_0 of $\mathfrak{b}(3, 1)$ can be isotoped to lie on Σ_2 as shown in the Schubert normal form (see Figure 3, center). Note that the geometric intersection number of m_i and l_i is one.

On the other hand, components of the link corresponding to the Heegaard diagrams in Figure 9 can be obtained by pushing l_i slightly into the handlebody corresponding to $\alpha = \{\alpha^*, m_1, m_2\}$ and pushing l_0 slightly into the handlebody corresponding to $\beta = \{\beta_1, m_0\}$, where m_0 is the meridian of $\mathfrak{b}(3, 1)$ on Σ_2. This can be seen explicitly if we redraw the Heegaard surface as in Figure 10. After isotoping unknot components, it is easy to see the link from these diagrams is equivalent to $\mathfrak{L}(3, 1)$. For general (u, v), the proof applies without change.   □

For integers u and v satisfying $-u < v < 0$, and where u and v odd, the corresponding diagram is obtained by reflecting the diagram of $\mathfrak{L}(u, -v)$ along a vertical line. Since $\mathfrak{L}(u, u - v) \cong \mathfrak{L}(u, -v)$, Heegaard diagrams for all $v \in (-u, u)$ with $\gcd(u, v) = 1$ and $(u, v) = (1, 0), (1, 1)$ are obtained from this approach. Such a diagram is called a *standard diagram* of $E(\mathfrak{L}(u, v))$.

A resolution of an intersection point of a meridian and a longitude on the Heegaard surface is called a *positive resolution* or a *negative resolution* when the meridian turns left or right, respectively, to the longitude in any direction; see Figure 11.
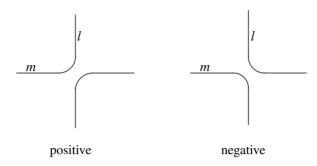
Figure 11: Positive and negative resolutions.

Corollary 7.7 *For $i = 1, 2$ suppose integers p_i and q_i satisfy $\gcd(p_i, q_i) = 1$ and $p_i > 0$. The Heegaard diagram $(\Sigma_2, \{\alpha_1, \alpha_2\}, \beta_1)$ of $E(K_0(u, v, p_1/q_1, p_2/q_2))$ is obtained in the following way: α_i is obtained by resolving intersection points of $|p_i|$ copies of m_i and $|q_i|$ copies of l_i positively or negatively if q_i is positive or negative, respectively. Especially when $(p_i, q_i) = (1, 0)$, the corresponding α_i is m_i.*

Proof This follows from the definition of Dehn surgery. Note that α_i is the meridian of the filling solid torus for $i = 1, 2$. □

Consider cyclic covers of the diagram of a 2–bridge knot $\mathfrak{b}(u, v)$ as shown in Figure 12. For $i \in \mathbb{Z}$, let $a_i = a_i(u, v)$ be a red strand connecting the left edge to the right edge and passing through $|i|$ copies of the fundamental domains, where the sign of i determines the direction of the strand; see Figure 12 for examples of strands. Let $A_i = A_i(u, v)$ be the set consisting of strands that can be isotoped into the neighborhood of $a_i(u, v)$ in the complement of basepoints. Some intersection points of $a_i(u, v)$ and β_1 can be removed by isotopy. Intersection points that cannot be removed are shown in Figure 12. Identifying endpoints of a_i, a diagram of a 2–bridge knot $\mathfrak{b}(U(u, v, i), V(u, v, i))$ can be obtained for some integers $U(u, v, i)$ and $V(u, v, i)$.

Let $a_* = a_*(u, v)$ and $a_\# = a_\#(u, v)$ be the strands in Figure 12. For $i = *, \#$, the set $A_i(u, v)$ and the functions $U(u, v, i)$ and $V(u, v, i)$ are defined similarly. For $i \in \mathbb{Z}$ or $i = *, \#$, consider $V(u, v, i) \in \mathbb{Z}/U\mathbb{Z} - \{0\}$ for $U = U(u, v, i) > 1$. When $U(u, v, i) = 1$, consider $V(u, v, i) \in \{0, 1\}$. In the latter case, we use the following conventions:

$$n \equiv \begin{cases} 1 & \text{if } n \text{ is odd modulo 1,} \\ 0 & \text{if } n \text{ is even modulo 1,} \end{cases} \quad \text{and} \quad \pm n \equiv \mp m \pmod{1} \text{ for } n \text{ odd and } m \text{ even.}$$
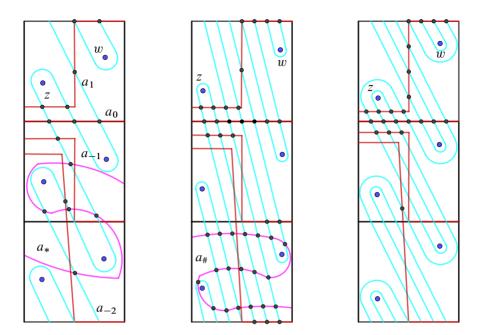
Figure 12: Cyclic covers of Heegaard diagrams corresponding to $(u, v) = (3, 1), (7, 1), (7, 2)$.

Lemma 7.8 *Suppose u and v, for u odd, are integers satisfying $(u, v) = (1, 0)$ or $0 < 2v < u$, and $\gcd(u, v) = 1$. For $i \in \{1, 0, -1, -2, *, \#\}$, the functions $U(u, v, i)$ and $V(u, v, i)$ can be expressed explicitly:*

(i) $U(u, v, 1) = u + 2v$ *and* $V(u, v, 1) = v$.

(ii) $U(u, v, 0) = u$ *and* $V(u, v, 0) = v$.

(iii) $U(u, v, -1) = u - 2v$ *and* $V(u, v, -1) \equiv v \pmod{u - 2v}$.

(iv) $U(u, v, -2) = |u - 4v|$ *and* $V(u, v, -2) \equiv v \operatorname{sign}(u - 4v) \pmod{|u - 4v|}$ *for* $u > 3$, *and* $U(3, 1, -2) = 1$ *and* $V(3, 1, -2) = 1$.

(v) $U(u, v, *) = 3u - 4v$ *and* $V(u, v, *) = u - v$.

(vi) $U(u, v, \#) = 3u - 2v$ *and* $V(u, v, \#) = 2u - v$.

Proof For fixed (u, v), let R_i and S_i be numbers of rainbows and stripes in the diagram of $\mathfrak{b}(U(u, v, i), V(u, v, i))$. Case (ii) is trivial, where $R_0 = v$ and $S_0 = u - 2v$. Suppose V' satisfies

$$0 < V' < U(u, v, i) \quad \text{and} \quad V' \equiv V(u, v, i) \pmod{U(u, v, i)}.$$

Define
$$\epsilon_i = \begin{cases} -1 & \text{if } 2V' < U(u, v, i), \\ 1 & \text{if } 2V' > U(u, v, i). \end{cases}$$

Then $(U(u, v, i), V(u, v, i))$ can be recovered from (R_i, S_i, ϵ_i) by

(9) $$U(u, v, i) = 2R_i + S_i \quad \text{and} \quad V(u, v, i) = \epsilon_i R_i.$$

Suppose that all isotopies on the surface move basepoints in the following discussion.

For cases (i) and (vi), let x_1 be the center of the fundamental domain and let $D_1 = N(x_1)$ be the neighborhood containing two basepoints z and w. Straightening strands isotopes the diagram by rotating D_1 clockwise and counterclockwise by π for cases (i) and (vi), respectively. Equivalently, the new β is obtained by pushing rainbows on the top edge to the bottom right and bottom left, respectively. Rainbows and stripes satisfy the following equations and we obtain the results by formulae in (9):

$$R_1 = R_0, \qquad S_1 = 2R_0 + S_0, \quad \epsilon_1 = 1,$$
$$R_\# = R_0 + S_0, \quad S_\# = 2R_0 + S_0, \quad \epsilon_\# = -1.$$

For case (v), let x_2 be the middle intersection point on the top edge and let $D_2 = N(x_2)$ be the neighborhood containing all rainbows. Straightening the strand isotopes the diagram by rotating D_2 clockwise by π. Then we have

$$R_* = R_0 + S_0, \quad S_* = S_0, \quad \epsilon_* = 1.$$

For case (iii), the number $U(u, v, -1)$ is the same as S_0. Straightening the strand isotopes the diagram by rotating D_2 counterclockwise, which induces the formula of $V(u, v, -1)$. This isotopy can also be regarded as pulling back rainbows once.

For case (iv), if $(u, v) = (3, 1)$, then the formula is obtained directly from Figure 12. If $u > 3$, then there are three subcases where $S_0 > 2R_0$, $2R_0 > S_0 > R_0$ and $R_0 > S_0$, equivalently $u > 4v$, $4v > u > 3v$ and $3v > u > 2v$, respectively. Note that u is odd, so $u \neq 4v$.

Suppose $S_0 > 2R_0$ (eg $(u, v) = (7, 1), (13, 3)$). In this subcase $V(u, v, -1) = v$. Straightening the strand isotopes the diagram by pulling back rainbows twice. Then $(U(u, v, -2), V(u, v, -2))$ is obtained by applying case (iii) twice, ie

$$U(u, v, -2) = u - 4v, \quad V(u, v, -2) \equiv v \pmod{u - 4v}.$$

Suppose $2R_0 > S_0 > R_0$ (eg $(u, v) = (7, 2), (15, 4)$). Straightening the strand isotopes the diagram by rotating D_2 counterclockwise by π. After isotopy, the number of

intersection points of a_{-2} and β is $U(u, v, -2) = 2R_0 - S_0 = 4v - u$. The number of rainbows is $R_{-2} = S_0 - R_0$ and $\epsilon_{-2} = -1$. Hence

$$V(u, v, -2) = U(u, v, -2) - (S_0 - R_0) = 7v - 2u.$$

Suppose $R_0 > S_0$ (eg $(u, v) = (7, 3)$). Straightening the strand isotopes the diagram by rotating D_2 counterclockwise by π. In this subcase, this isotopy is obtained by reversing the isotopy in case (v). Then

$$R_{-2} = R_0 - S_0, \quad S_{-2} = S_0, \quad \epsilon_{-2} = 1, \quad U(u, v, -2) = 4v - u, \quad V(u, v, -2) = 3v - u.$$

The formula for case (iv) then follows from summarizing the above subcases. $\qquad\square$

Remark 7.9 Indeed, for any $i \in \mathbb{Z}$, functions $U(u, v, i)$ and $V(u, v, i)$ might be expressed explicitly. For example, we have $U(u, v, i) = u + 2iv$ and $V(u, v, i) = v$ for $i > 0$. However, for $i < 0$, functions are more complicated so we omit the discussion.

The following lemma is a basic result from the Dehn surgery on the Hopf link.

Lemma 7.10 The manifold $M(u, v, p_1/q_1, p_2/q_2)$ is diffeomorphic to the lens space

$$L(p_1 p_2 - q_1 q_2, p_1 p_2' - q_1 q_2') \quad \text{where } p_2 q_2' - q_2 p_2' = -1.$$

Theorem 7.11 Suppose integers u_0 and v_0 satisfy $(u_0, v_0) = (1, 0)$ or $0 < 2v_0 < u_0$, and $\gcd(u_0, v_0) = 1$, where u is odd. Suppose $U_i = U(u_0, v_0, i)$ and $V_i = V(u_0, v_0, i)$. The knot $K_0 = K_0(u_0, v_0, p_1/q_1, p_2/q_2)$ is equivalent to $C(p, q, l, u, v)$ for (l, u, v) in [Table 3](#) and some (p, q). In cases (i)–(iv), $(p, q) = (p_1 p_2 - q_1 q_2, q_1)$. In cases (v)–(viii), $(p, q) = (p_1 p_2 - q_1 q_2, q_1 p_2)$. In cases (ix) and (x), $p = p_1 p_2 - q_1 q_2$ and $q \in \{\pm q_0^{\pm 1}\}$, where $q_0 = p_1 p_2' - q_1 q_2'$ is calculated in [Lemma 7.10](#).

Proof First, we make some comments on the parameters (p, q). [Lemma 7.10](#) provides a way to specify the ambient lens space of K_0. Explicitly, the lens space is $L(p, q)$, where $p = p_1 p_2 - q_1 q_2$ and $q \in \{\pm q_0^{\pm 1}\}$.

In cases (i)–(iv), $p_2 = 1$. Hence we can choose $q_2' = -1$ and $p_2' = 0$ in [Lemma 7.10](#). Then we can set $q_0 = p_1 p_2' - q_1 q_2' = q_1$. In cases (v)–(viii), $|q_1| = 1$. By [Proposition 7.3](#), we can switch the roles of (p_1, q_1) and (p_2, q_2) in [Lemma 7.10](#). So we can pick $p_1' = q_1$ and $q_1' = 0$ so that $p_1 q_1' - q_1 p_1' = -1$. Then we can set $q_0 = p_2 p_1' - q_2 q_1' = q_1 p_2$.

From [Remark 3.3](#) we know that $C(p, q, l, u, v)$ may be different from $C(p, q^{-1}, l, u, v)$. Hence to define a constrained knot, we need to fix the choice of q in the set $\{\pm q_0^{\pm 1}\}$.

case	conditions	$(l-1, u, v)$
(i)	$p_2 = 1, q_1 q_2 < 0$	$(-q_1 q_2, U_0, V_0)$
(ii)	$p_2 = 1, q_2 > 1, q_1 > p_1 > 0, U_{-1} \geq U_{-2}$	(p_1, U_{-1}, V_{-1})
(ii')	$p_2 = 1, q_2 > 1, q_1 > p_1 > 0, U_{-1} < U_{-2}$	$(q_1 q_2 - 2p_1, U_{-2}, V_{-2})$
(iii)	$p_2 = 1, q_2 < -1, -q_1 > p_1 > 0$	$(q_1 q_2 - 2p_1, U_*, V_*)$
(iv)	$(p_2, q_2) = (1, 0)$	$(0, U_0, V_0)$
(v)	$p_1 > 1, |q_1| = 1, q_1 q_2 < 0$	$(-q_1 q_2, U_0, V_0)$
(vi)	$p_1 > 1, q_1 = 1, p_2 > q_2 > 0$	$(p_1 p_2 - 2q_2, U_1, V_1)$
(vii)	$p_1 > 1, q_1 = -1, p_2 > -q_2 > 0$	$(-p_2, p_1 p_2 + 2q_2, U_\#, V_\#)$
(viii)	$(p_1, q_1) = (0, 1)$	$(0, U_{-1}, V_{-1})$
(ix)	$(p_2, q_2) = (1, 1), q_1 > 0, (p_1, q_1) \neq (1, 1)$	$(\pm q_1, U_n, V_n)$ for $n \in \mathbb{Z}$
(x)	$(p_2, q_2) = (1, -1), q_1 < 0, (p_1, q_1) \neq (1, -1)$	$l - 1 = \pm q_1$

Table 3: Cases where Dehn surgeries on magic links induce constrained knots.

For cases (i)–(viii), the later proof shows $q = q_0$. However, for cases (ix)–(x) it is hard to provide a general formula for the choice of q since the proof is not constructive.

We prove the theorem case by case:

For case (i), we consider two subcases:

(a) $p_2 = 1$ and $q_2 > 0, q_1 < 0$,

(b) $p_2 = 1$ and $q_2 < 0, q_1 > 0$.

The proofs of these two subcases are similar so we only prove case (a).

In case (a), $|q_2| = q_2$ and $|q_1| = -q_1$. Consider curves m_1, l_1, m_2 and l_2 in Figure 10 and the Heegaard diagram $(\Sigma_2, \{\alpha_1, \alpha_2\}, \beta_1)$ of $E(K_0)$ in Corollary 7.7. For example, if $q_2 = 3$, then α_2 is obtained by resolving intersection points of m_2 and three copies of l_2 positively. Let l_1' be the curve obtained by sliding l_1 over α_2 along an arc a around z; see Figure 13, top left. Let α_1' be obtained by taking $|p_1|$ copies of m_1 and $|q_1|$ copies of l_1' and resolving negatively. Then $(\Sigma_2, \{\alpha_1', \alpha_2\}, \beta_1)$ is also a Heegaard diagram of $E(K_0)$ since l_1' is isotopic to l_1 in the link complement. Consider the genus 1 surface Σ_1 obtained from Σ_2 by removing the 1–handle attaching to z and w. Then $(\Sigma_1, \alpha_1', \beta_1, z, w)$ is a doubly pointed Heegaard diagram of K_0. We can compare this diagram with the standard diagram of a constrained knot.

By construction, there are q_2 strands in l_1' connecting the left edge to the right edge, where (q_2-1) strands do not intersect the top edge and one strand intersects the top edge.
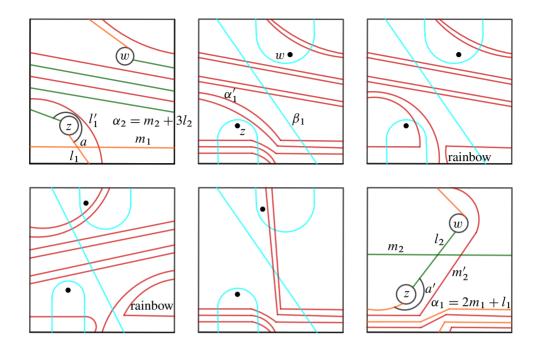
Figure 13: Examples of K_0.

Since m_1 is a strand connecting the left edge to the right edge, there are $(p_1 - q_1 q_2)$ strands in α_1' connecting the left edge to the right edge. These strands can be divided into two parts, in which the strands are isotopic to the strands a_0 and a_{-1} defined before Lemma 7.8, respectively. By counting the number of strands, we have

$$|A_0(u_0, v_0)| = p_1 \quad \text{and} \quad |A_{-1}(u_0, v_0)| = -q_1 q_2.$$

Hence $(\Sigma_1, \alpha_1', \beta_1, z, w)$ is the same as the standard diagram (see Figure 6) of

$$C(p_1 - q_1 q_2, q_1, -q_1 q_2 + 1, u_0, v_0).$$

Thus, the two knots are equivalent. For example, Figure 13, top middle, corresponds to $C(9, -2, 7, 3, 1) = C(9, 7, 7, 3, 1)$, where

$$(p_1, q_1, p_2, q_2, u, v) = (3, -2, 1, 3, 3, 1).$$

Cases (ii)–(iv) are proven by a similar strategy. Indeed, we can compare α_1' in the doubly pointed diagram of K_0 with the standard diagram of a constrained knot to obtain the parameters. In particular, the type and the number of strands in α_1' are important, so we only state the main difference about the curve α_1'.

For cases (ii) and (ii′), let α_1' be the curve as defined in case (i). It is the union of strands with endpoints on the left edge and the right edge. By the assumption $q_1 > p_1$, we may have rainbows in α_1', ie strands whose endpoints are on the same edge. Since the rainbows on the right edge do not bound a basepoint, we can isotopy α_1' to remove them. After removing p_1 rainbows on the right edge, there are $(q_1 q_2 - 2p_1)$ strands and p_1 strands isotopic to a_{-1} and a_{-2}, respectively:

$$|A_{-1}| = q_1 q_2 - 2p_1 \quad \text{and} \quad |A_{-2}| = p_1.$$

The choice of case (ii) and case (ii′) depends on if $U_{-1} \geq U_{-2}$ or $U_{-1} < U_{-2}$, respectively. This is because the parameter u is the greater number in $\{U_{-1}, U_{-2}\}$.

For case (iii), the pair of sets (A_{-1}, A_{-2}) in the above proof is replaced by (A_{-1}, A_*). Counting the number of strands, we have

$$|A_{-1}| = q_1 q_2 - 2p_1 \quad \text{and} \quad |A_*| = p_i.$$

By Lemma 7.8, the number U_* is always greater than U_{-1}.

For case (iv), all strands are isotopic to a_0.

Examples can be found in Figure 13. In all examples, $(u, v) = (3, 1)$. In the top right subfigure, the diagram of K_0 is in case (ii) with $(p_1, q_1, p_2, q_2) = (1, 2, 1, 3)$, which corresponds to $C(-5, 2, 2, 1, 0) = C(5, 3, 2, 1, 0)$. In the bottom left subfigure, the diagram of K_0 is in case (iii) with $(p_1, q_1, p_2, q_2) = (1, -2, 1, -3)$, which corresponds to $C(-5, -2, 5, 5, 2) = C(5, 2, 5, 5, 2)$. In the bottom middle subfigure, the diagram of K_0 is in case (iii) with $(p_1, q_1, p_2, q_2) = (3, -2, 1, 0)$, which corresponds to $C(3, -2, 1, 3, 1) = C(3, 1, 1, 3, 1)$.

For proofs of cases (v)–(viii), we consider the curve m_2' obtained by sliding m_2 over α_1 along an arc a' around z; see Figure 13, bottom right, for $p_1 = 2$. Now the resulting diagram of $E(K_0)$ is $(\Sigma_2, \{\alpha_1, \alpha_2'\}, \beta_1)$, where α' is obtained from m_2' and l_2 by resolution. The proofs are similar to cases (i)–(iv).

For cases (ix) and (x), the diagrams are more complicated. By Proposition 1.1, we can check by the distribution of the spin^c structures of intersection points that the knot K_0 is a constrained knot. The parameter l can be obtained by counting the number of strands. $\qquad \square$

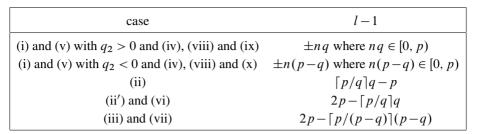The following corollary is obtained by changing parameters in Table 3.

case	$l-1$
(i) and (v) with $q_2 > 0$ and (iv), (viii) and (ix)	$\pm nq$ where $nq \in [0, p)$
(i) and (v) with $q_2 < 0$ and (iv), (viii) and (x)	$\pm n(p-q)$ where $n(p-q) \in [0, p)$
(ii)	$\lceil p/q \rceil q - p$
(ii′) and (vi)	$2p - \lceil p/q \rceil q$
(iii) and (vii)	$2p - \lceil p/(p-q) \rceil (p-q)$

Table 4: Choices of the parameter l.

Corollary 7.12 *Suppose integers p and q satisfy $p > q > 0$ and $\gcd(p,q) = 1$. The choices of l from Theorem 7.11 are in Table 4. Note that Theorem 1.8 follows from the first two rows in Table 4.*

Remark 7.13 For integers u_0 and v_0 satisfying $(u_0, v_0) = (1, 1)$ or $0 < -2v_0 < u_0$, and $\gcd(u_0, v_0) = 1$, where u_0 is odd, the surgery description can be induced similarly to Table 3. We omit the explicit description.

We describe some special examples of Table 3 as follows.

Consider integers u_0 and v_0 satisfying $(u_0, v_0) = (1, 0)$ in Theorem 7.11. We know that the manifold $E(\mathfrak{L}(1,0))$ is diffeomorphic to $S^1 \times F$, where F is a disk with two holes. For integers p_1, p_2, q_1 and q_2 satisfying $p_1 p_2 \neq q_1 q_2$, the knot $K_0(1, 0, p_1/q_1, p_2/q_2)$ is a torus knot in a lens space.

Cases (iii) and (vii) in Table 3 cover the cases $(u, v) = (3, \pm 1)$. By Corollary 7.12, for $p, q \in \mathbb{Z}$ with $p > q > 0$, the knot $C(p, \pm q, 2p - \lceil p/q \rceil q + 1, 3, \pm 1)$ is a torus knot.

Theorem 7.14 *The knot $C(p, q, 1, u, v)$ is the connected sum of the 2–bridge knot $\mathfrak{b}(u, v)$ and the core knot $C(p, q, 1, 1, 0)$ of $L(p, q')$, where $qq' \equiv 1 \pmod{p}$.*

Proof By case (iv) in Theorem 7.11, the knot $C(p, q, 1, u, v)$ is identified with $K_0(u, v, p/q, 1/0)$, which is obtained by the p/q surgery on the meridian of $\mathfrak{b}(u, v)$. By Corollary 3.9, the knot $C(p, q, 1, 1, 0)$ is the core knot, which is obtained by the p/q surgery on one component of the Hopf link. $\qquad\square$

8 1–Bridge braid knots

In this section we describe another approach to construct constrained knots by Dehn surgeries. Many results are based on [15, Section 3]. The main objects in this section are 1–bridge braids, defined below.

Definition 8.1 A knot in the solid torus $S^1 \times D^2$ is called a 1–*bridge braid* if it is isotopic to a union of two arcs $\gamma \cup \delta$ such that $\gamma \subset \partial(S^1 \times D^2)$ is braided, meaning it is transverse to each meridian $\{\text{pt}\} \times \partial D^2$, and δ is a bridge, meaning it is properly embedded in some meridional disk $\{\text{pt}\} \times D^2$.

We denote 1–bridge braids by $B(w, b, t)$ [13], where $w > 0$ is the *winding number*, $b \in [0, w-2]$ is the *bridge width*, and $t \in [1, w-1]$ is the *twist number*. When $b = 0$, the 1–bridge braid can be isotoped to lie on $\partial(S^1 \times D^2)$. Let $B(w, w-1, t)$ denote $B(w, 0, t+1)$.

As mentioned in [15, Section 3], after isotopy, the arc γ can be lifted to a straight line (a geodesic) in the universal cover $\mathbb{R}^2$ of $\partial(S^1 \times D^2)$, which is still denoted by γ. Suppose that γ connects $(0, 0)$ to (t', w), where $t' \in \mathbb{Q} \cap [t, t+1]$. Let $B(w, s(\gamma))$ denote this 1–bridge braid, where $s(\gamma) = t'/w$ is called the *inverse slope* of γ. Suppose $s = n/d$ with $\gcd(n, d) = 1$. Suppose that the integer $n_i \in [0, d)$ satisfies $n_i \equiv ni \pmod{d}$. Then b is given by the formula

$$b = \#\{i \in [1, w-1] \mid n_i < n_w\}.$$

Definition 8.2 Suppose integers p and q satisfy $0 < q < p$ and $\gcd(p, q) = 1$. Denote the knot in $L(p, q)$ obtained by Dehn filling $(S^1 \times D^2, B(w, s(\gamma)))$ along the curve on $\partial(S^1 \times D^2)$ with slope p/q by $B(w, s(\gamma), p, q)$. This is called a 1–*bridge braid knot*.

Proposition 8.3 *For a 1–bridge braid $B(w, s(\gamma))$, suppose s represents the core of the solid torus, and suppose t represents the meridian of the braid. For $j \in [1, w-1]$, define*

$$\theta_j = \begin{cases} 1 & \text{if } n_j < n_w, \\ 0 & \text{if } n_j > n_w. \end{cases}$$

Then the 2–variable Alexander polynomial of $B(w, s(\gamma))$ is

$$\Delta(s, t) = \sum_{i=0}^{w-1} s^i t^{\sum_{j=1}^{i} \theta_j}.$$

Proof Suppose $H_2 = S^1 \times D^2 - N(\delta)$, which is diffeomorphic to a genus two handlebody. Let D be the canceling disk of δ. There are two meridian disks $\{\text{pt}\} \times \partial D^2$ and D of H_2. Suppose their boundaries are α_1 and α_2, respectively, then suppose that $\beta = \partial N(\gamma)$ and $\Sigma = \partial H_2$. Then $(\Sigma, \{\alpha_1, \alpha_2\}, \beta)$ is a Heegaard diagram of

Figure 14: Left: 1–bridge braid in $\mathbb{R}^2$. Right: simple intervals.

$S^1 \times D^2 - N(B(w, s(\gamma)))$. It induces a presentation of the fundamental group by the method in Section 6:

$$\pi_1(S^1 \times D^2 - N(B(w, s(\gamma)))) = \langle s, t \mid \omega t \omega^{-1} t^{-1} = 1 \rangle,$$

where $\omega = s t^{\theta_1} s t^{\theta_2} s \cdots s t^{\theta_{w-1}} s$. Then we can calculate the Alexander polynomial by Fox calculus [42, Chapter II].   □

Let F_n be the nth Farey sequence, ie the sequence containing all rational numbers x/y with $0 \leq x \leq y \leq n$ and $\gcd(x, y) = 1$, listed in the increasing order. For example:

$$(10) \quad F_1 = \left(\tfrac{0}{1}, \tfrac{1}{1}\right), \; F_2 = \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right), \; F_3 = \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right), \; F_4 = \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right).$$

For a fixed integer w, suppose f_- and f_+ are successive terms in F_{w-1}. For any two 1–bridge braids with inverse slopes $s_2, s_2 \in (f_-, f_+)$ there is an isotopy between them [15, Section 3]. If $s(\gamma) \in (f_-, f_+)$, the interval $\mathbb{S}(\gamma) = [f_-, f_+]$ is called the *simple interval* of γ. Two examples are shown in Figure 14, left.

For integers w and t satisfying $\gcd(w, t) = 1$, the 1–bridge braid knot $B(w, t/w, p, q)$ is the (w, t) torus knot in $L(p, q)$ defined in Section 2. Suppose $f_\pm = n_\pm / d_\pm$, where $n_\pm$ and $d_\pm$ are integers satisfying $\gcd(n_\pm, d_\pm) = 1$. If $d_\pm \mid w$, then the 1–bridge braid knot $B(w, s(\gamma), p, q)$ with $s(\gamma) \in (f_-, f_+)$ is the $(1, \mp w/d_\pm)$ cable knot of the $(d_\pm, nw/d_\pm)$ torus knot in $L(p, q)$, respectively; see [15, Section 3.1]. The braids $B(\omega, s(\gamma))$ in the above two cases are called *torus braids* and *cable braids*, respectively. In other cases, the braid $B(w, s(\gamma))$ is called a *strict braid*.

Theorem 8.4 *The 1–bridge braid knot $B(w, s(\gamma), p, q)$ is a simple knot if and only if $q/p \in \mathbb{S}(\gamma)$. In this case, it is the simple knot $S(p, q, wq)$.*

Proof The sufficient part follows from the discussion before [15, Theorem 3.2]. Indeed, the arc γ can be isotoped to have the inverse slope q/p (if $q/p = f_\pm$, then let the slope of γ be $f_\pm \mp \epsilon$ for small $\epsilon > 0$). Then the knot is the union of two arcs of slopes 0 and q/p, respectively. Then it is straightforward to check that the knot is a simple knot. Note that the knot is homologous to wq of the core of the filling solid torus. Thus, the knot is $S(p, q, wq)$.

The necessary part for a strict braid is shown by [15, Theorem 3.2]. When $B(w, s(\gamma))$ is not strict, the proof of [15, Theorem 3.2] still applies because $d_\pm < w$. $\square$

Let us consider special cases of simple knots obtained from Theorem 8.4. Consider examples of Farey sequences in (10). For $w \leq 3$, all simple knots are from torus braids. For $w \leq 4$, all simple knots are from either torus braids or cable braids. For $w \geq 4$, the union of the simple intervals for torus braids and cable braids are shown in Figure 14, right, where red arcs represent torus braids (they are Berge–Gabai knots of type I; see [12; 3; 2]), blue arcs represent $(1, \pm 2)$ cable braids (they are Berge–Gabai knots of type II), and green arcs represent other cable braids.

Proof of Theorem 1.9 By Theorem 8.4, simple knots are 1–bridge braid knots. For constrained knots that are not simple knots, we show $C(p, q, l, u, 1)$ is equivalent to a 1–bridge braid knot. The case $C(p, q, l, u, -1) = C(p, q, l, u, u-1)$ is the mirror image of $C(p, -q, l, u, 1)$ by Proposition 3.4 so is also equivalent to a 1–bridge braid knot.

The proof is inspired by Figure 15, left. Suppose $(T^2, \alpha_1, \beta_1, z, w)$ is the standard diagram of $C(p, q, l, u, 1)$. By definition, the constrained knot is the union of two arcs a and b connecting z to w in $T^2 - \alpha_1$ and $T^2 - \beta_1$, pushed slightly into the α_1–handlebody and the β_1–handlebody, respectively. The arc a can be chosen as a horizontal one, and there are infinitely many choices of isotopy classes of b on T^2. Let γ_i denote different choices of b for $i \in \mathbb{Z}$. All choices induce equivalent knots because they are isotopic in the β_1–handlebody.

Since there is only one rainbow for β_1, the arc γ_i does not have any rainbows. For a large integer i, the arc γ_i can be isotoped to a straight line. Then γ_i is transverse to each meridian disk of the α_1–handlebody and the union of a and γ_i is a 1–bridge braid in the α_1–handlebody. Hence $C(p, q, l, u, 1)$ is equivalent to a 1–bridge braid knot. $\square$
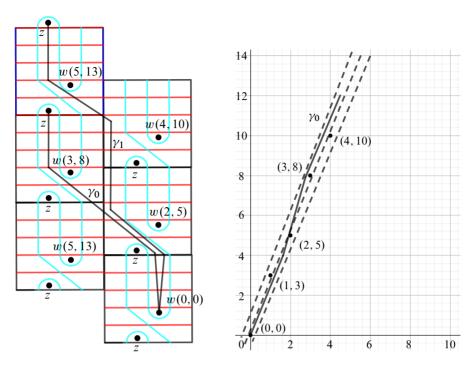
Figure 15: Arcs γ_i for $C(5, 3, 2, 3, 1)$ (left) and γ_0 for $C(5, 3, 2, 3, 1)$ in $\mathbb{R}^2$ (right).

It is possible to find the explicit formula of $B(w(\gamma_i), s)$ in Theorem 1.9 as follows:

Suppose lifts of w in the universal cover $\mathbb{R}^2$ of T^2 are lattice points of $\mathbb{Z}^2$ as in Figure 15, right. Then domains in Figure 15, left, lie in the narrow bands with dotted boundaries in Figure 15, right. From the parametrization of the constrained knot, we know $C(p, q, l, u, 1)$ is in $L(p, q')$, where $qq' \equiv 1 \pmod{p}$. Then the slope of the dotted boundaries is p/q'. Indeed, these boundaries are β_0 in the standard diagram (T^2, α_0, β_0) of $L(p, q')$.

Suppose
$$\lambda = \frac{qq' - 1}{p} \quad \text{and} \quad \epsilon = \begin{cases} 0 & \text{if } l + q \le p, \\ 1 & \text{if } l + q > p. \end{cases}$$

Suppose γ_0 is the first arc that can be straightened in the lift of $T^2 - \beta_1$. Suppose D_j for $j \in \mathbb{Z}/p\mathbb{Z}$ are regions in the new diagram C mentioned in Section 3. The part of γ_i that lies in $\bigcup_{i=l+1}^{p} D_i$ and the disk bounded by the unique rainbow of β_1 around a basepoint is called the part *in the generalized rainbow*. There are two parts of γ_i in generalized rainbows related to z and w.

The parameter $w(\gamma_i)$ is the same as $|\gamma_i \cap \alpha_1|$. Thus

$$w(\gamma_i) = p(u-3) + 2(p-l+1) + (q+l-1-p\epsilon) + pi = p(u-1-\epsilon+i) + q - l + 1,$$

where $p(u-3) + 2(p-l+1)$ is from parts of γ_i in the two generalized rainbows and $(q+l-1-p\epsilon) + pi$ is from the remaining part. Any lift of w in the left band in Figure 15 has the coordinates

$$(\lambda + nq', q + np) \quad \text{for some } n \in \mathbb{Z}.$$

The closest lift of w near γ_i other than $(0,0)$ has the coordinates

$$(\lambda + n_0 q', q + n_0 p) \quad \text{where } n_0 = (u-1)/2 - \epsilon + i.$$

It lies at a lift of the region D_l that intersects the endpoint of the part of γ_i in the generalized rainbow related to z. Thus, the inverse slope of γ_i is

$$\frac{\lambda + n_0 q'}{q + n_0 p} - r$$

for a small rational number r.

In practice, for given parameters $(p, q, l, u, 1)$, it is possible to determine if a constrained knot $C(p, q, l, u, 1)$ is from a torus braid or a cable braid. For example, consider $(l, u, v) = (2p - \lceil p/q \rceil q + 1, 3, 1)$ and $i = 0$. Then

$$\epsilon = 1, \quad n_0 = 0 \quad \text{and} \quad \omega = \left(1 + \left\lceil \frac{p}{q} \right\rceil\right)q - p.$$

The inverse slope is $\lambda/q - r$. Suppose $x = (1 + \lceil p/q \rceil)\lambda - q'$. Since

$$\frac{\lambda}{q} = \frac{x + q'}{w + p},$$

the rational number x/w is in the simple interval $\mathbb{S}(\gamma_0)$, ie γ_0 is isotopic to the arc with inverse slope x/w. Thus $C(p, q, 2p - \lceil p/q \rceil q + 1, 3, 1)$ is a torus knot. This is consistent with the example from the magic link $\mathfrak{L}(1, 0)$ mentioned in Section 7.

9 SnapPy manifolds

A compact orientable manifold M with torus boundary is called a (*hyperbolic*) 1–*cusped manifold* if the interior of M admits a hyperbolic metric of finite volume. All 1–cusped manifolds that have ideal triangulations with at most nine ideal tetrahedra are included in *SnapPy* [7]. They are called *SnapPy manifolds*. In this section we explain

the strategy used to study the relation between 1–cusped manifolds and constrained knots by computer program. Code and results can be found in [45].

Suppose M is a 1–cusped manifold and suppose γ is a simple closed curve on ∂M. The pair (M, γ) is called an *exceptional filling* if Dehn filling along γ gives a nonhyperbolic manifold $M(\gamma)$. For such (M, γ), the core of the filling solid torus induces a knot in $M(\gamma)$. The induced knot $K(M, \gamma)$ is called a *SnapPy knot* if M is a SnapPy manifold. Dunfield provided a census of exceptional fillings for SnapPy manifolds [9]. In this census, there are 44 487 exceptional fillings (M, γ), covering 38 056 different SnapPy manifolds, for which $M(\gamma)$ is a lens space.

Suppose $M(\gamma) = L(p, q)$ and m is the meridian of $K = K(M, \gamma)$. If $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ and it is generated by t, then $\tau(K) = \Delta_K(t)/(1 - t)$ [42]. The Alexander polynomial only depends on M and can be found in *SnapPy*. The Euler characteristic $\chi(\widehat{HFK}(M(\gamma), K))$ can be calculated by Lemma 5.7. Suppose it is $\sum a_i t^i$. Since

$$H_1(M; \mathbb{Z})/([m]) \cong H_1(M(\gamma); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z},$$

we know $[m] = t^p$. Then $\chi(\widehat{HFK}(M(\gamma), K))$ can be decomposed into p polynomials

$$\sum_{i \equiv i_0 \ (\mathrm{mod}\ p)} a_i t^i \quad \text{for } i_0 \in [0, p).$$

Suppose

$$F_{i_0}(t) = \sum_{i \equiv i_0 \ (\mathrm{mod}\ p)} a_i t^{(i - i_0)/p}$$

and let $f_i(t)$ be images of $F_i(t)$ in $\mathbb{Z}[t]/\pm (t)$. The exceptional filling (M, γ) has n *form(s)* if the set $\{f_i(t) \mid i \in [0, p)\}$ has n elements.

If $F_i(t)$ is a monomial for any i, then (M, γ) has 1 form. By Theorem 1.6, the Euler characteristic must be the same as the simple knot in the same homology class. Such an (M, γ) is called a *simple filling*. It does not necessarily induce a simple knot since Conjecture 1.7 has not been proven yet.

For $l = 1$, the constrained knot $C(p, q, l, u, v)$ is not hyperbolic since it is satellite by Theorem 7.14. If $F_i(t)$ is symmetric, coefficients of $F_i(t)$ are alternating for any i, and (M, γ) has 2 forms, then K might be a constrained knot $C(p, q', l, u, v)$, where $l > 1$, $u > 1$ and $q' \equiv \pm q^{\pm 1} \pmod{p}$. As in the proof of the necessary part of Theorem 1.2, a tuple of *virtual parameters* (l, u, v) can be calculated by $F_i(-1)$. Conversely, given (p, q', l, u, v), the characteristic of the corresponding constrained knot is given by Theorem 4.6. If $\chi(\widehat{HFK}(K))$ is equivalent to $\chi(\widehat{HFK}(C(p, q', l, u, v)))$ as elements

in $\mathbb{Z}[H_1(M;\mathbb{Z})]$ for virtual parameters (l, u, v), then (M, γ) is called a *constrained filling*. If symmetrized Alexander polynomials of K and $C(p, q', l, u, v)$ are the same, then (M, γ) is called a *general constrained filling*. If $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, then (M, γ) is a constrained filling if and only it is a general constrained filling.

If Tors $H_1(M;\mathbb{Z})$ is nontrivial, then the Turaev torsion $\tau(M)$ can be calculated by a presentation of $\pi_1(M)$. *SnapPy* provides a presentation of $\pi_1(M)$ and the related words of the preferred meridian and the preferred longitude (they are not necessarily the same as the meridian and the longitude mentioned in Section 2). By the filling slope from Dunfield's census, the homology class $[m] \in H_1(M;\mathbb{Z})$ is obtained. The algorithm described above also works and definitions also apply to this case.

The code in [45] constructs complements of constrained knots in *SnapPy* by functions in the Twister package. Then the function $M.identify()$ in *SnapPy* tells us if the manifold with a constrained filling is indeed the complement of a constrained knot. Mirror manifolds are not distinguished here.

In Dunfield's census, there are 16 355 simple fillings and 8537 constrained fillings, covering 15 262 and 8508 SnapPy manifolds, respectively. All 15 262 and 8421 of 8508 SnapPy manifolds are complements of simple knots and constrained knots, respectively. There are 1838 manifolds that are both complements of simple knots and constrained knots with $u > 1$. Thus, there are 21 845 SnapPy manifolds that are complements of constrained knots in lens spaces. Other than these manifolds, there are 77 SnapPy manifolds that are complements of 2–bridge knots, which are special cases of constrained knots.

The choice of the slope in a constrained filling is subtle. For example, suppose $M = m003$, and $\gamma_1 = (-1, 1)$ and $\gamma_2 = (0, 1)$ in the basis from *SnapPy*. Then both $M(\gamma_1)$ and $M(\gamma_2)$ are diffeomorphic to $L(5, 4)$ and M is the complement of $C(5, 4, 5, 3, 1)$. Indeed, there is an isometry of M sending γ_1 to γ_2. Both $M(\gamma_1)$ and $M(\gamma_2)$ induce the same knot, $C(5, 4, 5, 3, 1)$. All nine pairs of slopes in Dunfield's census with this subtlety are from isometries, except the case where $M = m172$, $\gamma_1 = (0, 1)$ and $\gamma_2 = (1, 1)$. Manifolds $M(\gamma_1)$ and $M(\gamma_2)$ are oppositely oriented copies of the same lens space. The first slope induces $S(49, 18, 7)$ and the second induces $S(49, 18, 21)$ (up to mirror image), which are not equivalent. This example is interesting in the study of cosmetic surgery [4]. To summarize: the SnapPy knots induced by $15\,262 + 8421 = 23\,683$ constrained fillings in the above discussion are all constrained knots.

There are 87 SnapPy manifolds with constrained fillings that are not complements of constrained knots. For such a manifold either the constrained knot with corresponding virtual parameters is not hyperbolic, or there is another SnapPy manifold which is the complement of the constrained knot with the same parameters. For example, the manifold $m390$ has a constrained filling $(1, 0)$ with virtual parameters $(7, 4, 7, 5, 2)$, while $E(C(7, 4, 7, 5, 2))$ is diffeomorphic to $s090$.

If Tors $H_1(M; \mathbb{Z})$ is nontrivial, then there are 54 general constrained fillings that are not constrained fillings. For example, manifolds $M_1 = m400$ and $M_2 = m141$ satisfy

$$|\text{Tors } H_1(M_i; \mathbb{Z})| = 2 \quad \text{and} \quad \Delta_{M_i}(t) = t^5 - t^4 + t^2 + t^{-2} - t^{-4} + t^{-5} \quad \text{for } i = 1, 2,$$

and $M_1(1,1) \cong M_2(-1, 1) \cong L(18, 13)$. Both manifolds have general constrained fillings, and $M_2 \cong E(C(18, 3, 18, 3, 1))$. Calculation shows $(M_1, (1, 1))$ is not a constrained filling, ie the Euler characteristic of the induced knot is different from that of $C(18, 3, 18, 3, 1)$.

For the exceptional manifolds in Proposition 1.10, manifolds $m206$ and $m370$ have exceptional fillings with 2 forms and have virtual parameters $(l, u, v) = (5, 5, 2)$ and $(8, 5, 2)$, respectively. Unfortunately, both exceptional fillings are not even general constrained fillings. The manifold $m390$ is discussed above. For other 5–manifolds there is no lens space filling (or even $S^1 \times S^2$ filling). It is harder to obtain information in Heegaard Floer theory.

In the rest of this section we discuss the ways to obtain the genus and the fiberness of a knot. The genera and fiberness of Snappy knots can also be found in [45].

Definition 9.1 Suppose K is a knot in $Y = L(p, q)$ and suppose

$$H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \cong \langle t, r \rangle (dr).$$

By the excision theorem, Poincaré duality and the universal coefficient theorem,

$$H_2(Y, K; \mathbb{Z}) \cong H_2(E(K), \partial E(K); \mathbb{Z}) \cong H^1(E(K); \mathbb{Z}) \cong \text{Hom}(H_1(E(K); \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}.$$

Suppose S is a connected, oriented and properly embedded surface representing the generator of $H_2(E(K), \partial E(K); \mathbb{Z})$. It is called a *Seifert surface* of K. Let the *genus* $g(K)$ and the *Thurston norm* $x([S])$ be the minimum values of $g(S)$ and $-\chi(S)$ among all Seifert surfaces, respectively.

Definition 9.2 For a homogeneous element x of $\widehat{HFK}(Y, K)$, suppose

$$\text{gr}(x) = at + br \in H_1(E(K); \mathbb{Z}).$$

Let $\text{gr}_0(x)$ be the number a. The width of $\widehat{HFK}(Y, K)$ is the maximum value of $|\text{gr}_0(x) - \text{gr}_0(y)|$ among all pairs of homogeneous elements (x, y). Suppose homogeneous elements x_0 and y_0 satisfy

$$\text{width } \widehat{HFK}(Y, K) = |\text{gr}_0(x_0) - \text{gr}_0(y_0)|.$$

Suppose $H(x_0)$ is the subgroup of $\widehat{HFK}(Y, K)$ generated by homogeneous elements x satisfying $\text{gr}_0(x) = \text{gr}_0(x_0)$. The *top rank* of $\widehat{HFK}(Y, K)$ is $\dim_{\mathbb{Q}} H(x_0) \otimes \mathbb{Q}$.

Theorem 9.3 [21; 27] *Consider* Y, K *and* S *as in* Definition 9.1 *such that* $E(K)$ *is irreducible. Suppose* m *is the meridian of* K. *Then the width of* $\widehat{HFK}(Y, K)$ *equals* $x([S]) + |[m] \cdot [\partial S]|$, *where* $[m] \cdot [\partial S]$ *is the algebraic intersection number on* $\partial E(K)$.

Proposition 9.4 *Let* Y, K *and* S *be as in* Definition 9.1. *Suppose* $E(K)$ *is irreducible. Suppose* (m, l) *is the regular basis of* K. *Let* n *be the minimum number of boundary components of a Seifert surface. Then* $|[m] \cdot [\partial S]| = p/d$ *and* $n = \gcd(d, p/d)$. *Thus,*

$$x([S]) = \text{width}(\widehat{HFK}(Y, K)) - \frac{p}{d} \quad \text{and} \quad g(K) = 1 + \frac{1}{2}\left(x([S]) - \frac{p}{d}\right).$$

Proof Suppose $[K] = k[b]$, where $[b]$ is a generator of $H_1(Y; \mathbb{Z})$. Since $d = \gcd(p, k)$, the order of $[K]$ in $H_1(Y; \mathbb{Z})$ is p/d. By Poincaré duality and the universal coefficient theorem, we have

$$H_2(E(K); \mathbb{Z}) \cong H^1(E(K), \partial E(K); \mathbb{Z}) \cong \text{Hom}(H_1(E(K), \partial E(K)), \mathbb{Z}) = 0.$$

By the long exact sequence from $(E(K), \partial E(K))$, the boundary map

$$\partial_* : H_2(E(K), \partial E(K); \mathbb{Z}) \to H_1(\partial E(K); \mathbb{Z})$$

is injective and the image of ∂_* is the same as the kernel of the map

$$i_* : H_1(\partial E(K); \mathbb{Z}) \to H_1(E(K); \mathbb{Z}).$$

Since $H_1(E(K); \mathbb{Z})/([m]) \cong H_1(Y; \mathbb{Z})$, we have $[\partial S] = \pm(x[m] + p/d [l])$ for some $x \in \mathbb{Z}$. Then $|[m] \cdot [\partial S]| = p/d$ and $n = \gcd(x, p/d)$.

Let $[m]$ and $[l]$ also denote their images in $H_1(E(K); \mathbb{Z})$. By Lemma 5.2, we have

$$[m] = \pm\left(\frac{p}{d}\right)t + ar \quad \text{and} \quad \gcd\left(\frac{p}{d}, d, a\right) = 1.$$

Suppose $[l] = yt + zr$ for some $y, z \in \mathbb{Z}$. Since $[\partial S] \in \mathrm{Ker}(i_*)$, we know $xa + (p/d)z$ is divisible by d. Suppose $n_0 = \gcd(d, p/d)$. Then $\gcd(n_0, a) = 1$ and $n_0 \mid xa + (p/d)z$. Thus $n_0 \mid x$ and $n_0 \mid n$. Suppose l^* is the homological longitude. Then $n[l^*] = [\partial S]$ and the image of $[l^*]$ in $H_1(E(K); \mathbb{Z})$ is wr for some $w \in \mathbb{Z}$. Thus $n \mid d$ and $n \mid n_0$. This induces $n = n_0$. $\qquad\square$

Theorem 9.5 [21; 26] *Consider Y, K and S as in Definition 9.1 such that $E(K)$ is irreducible. If the top rank of $\widehat{HFK}(Y, K)$ is 1, then K is fibered with the fiber S.*

Proof Suppose $Y(S)$ is the balanced sutured manifold (N, ν), where $N = Y - \mathrm{Int}(S \times I)$ and $\nu = \partial S \times I$. Lemma 3.9 and the proof of Theorem 1.5 in [21] imply that the rank of $SFH(Y(S))$ is the same as the top rank of $\widehat{HFK}(Y, K)$. Then $Y(S)$ is a product sutured manifold by [21, Theorem 9.7], which implies K is fibered with fiber S. $\qquad\square$

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