Constrained knots in lens spaces

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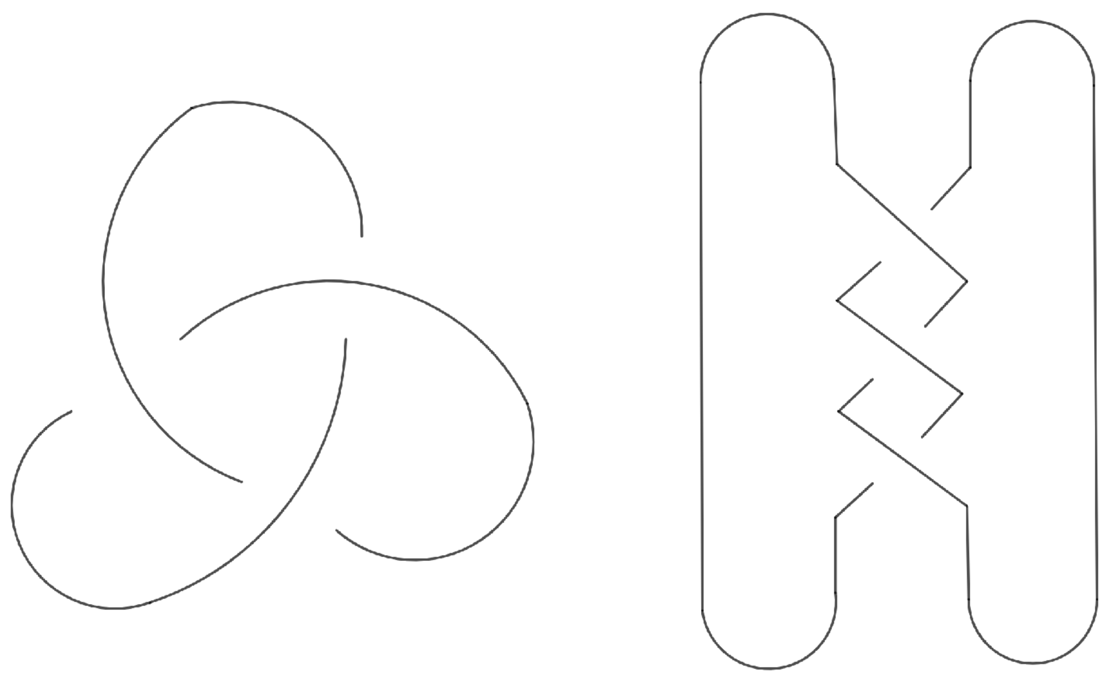
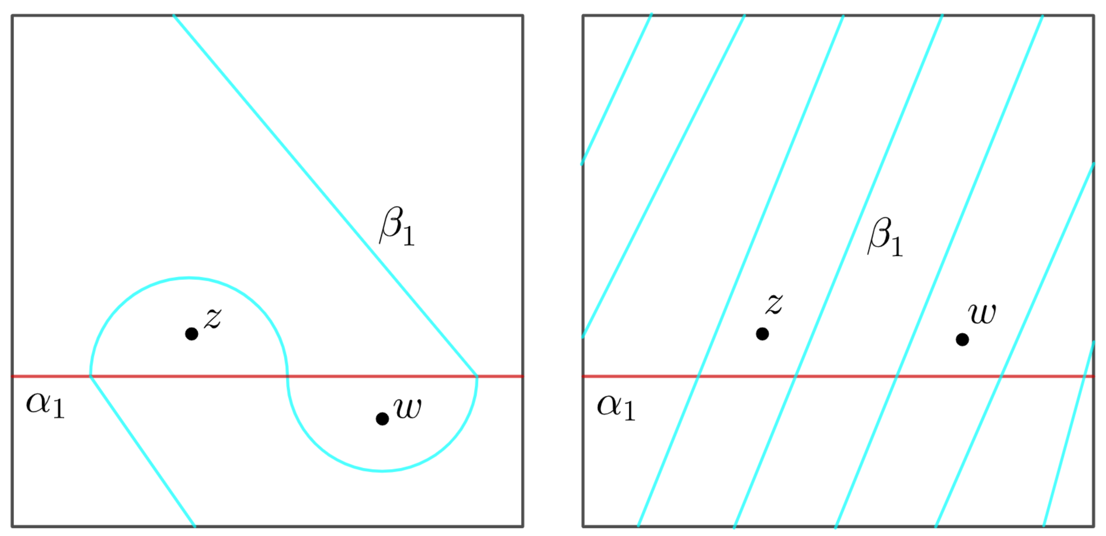
Motivation

- Constrained knots are generalizations of 2-bridge knots in S^3 and simple knots in lens spaces (see the following examples).
- Kronheimer and Mrowka conjectured that for any knot K in any closed 3-manifold, the knot Floer homology $\widehat{HFK}(Y,K)$ with $\mathbb{C}$ cofficient is isomorphic to the instanton knot homology $KHI(Y,K)$. It is known that this conjecture holds for all alternating knots, which include all 2-bridge knots.

Examples



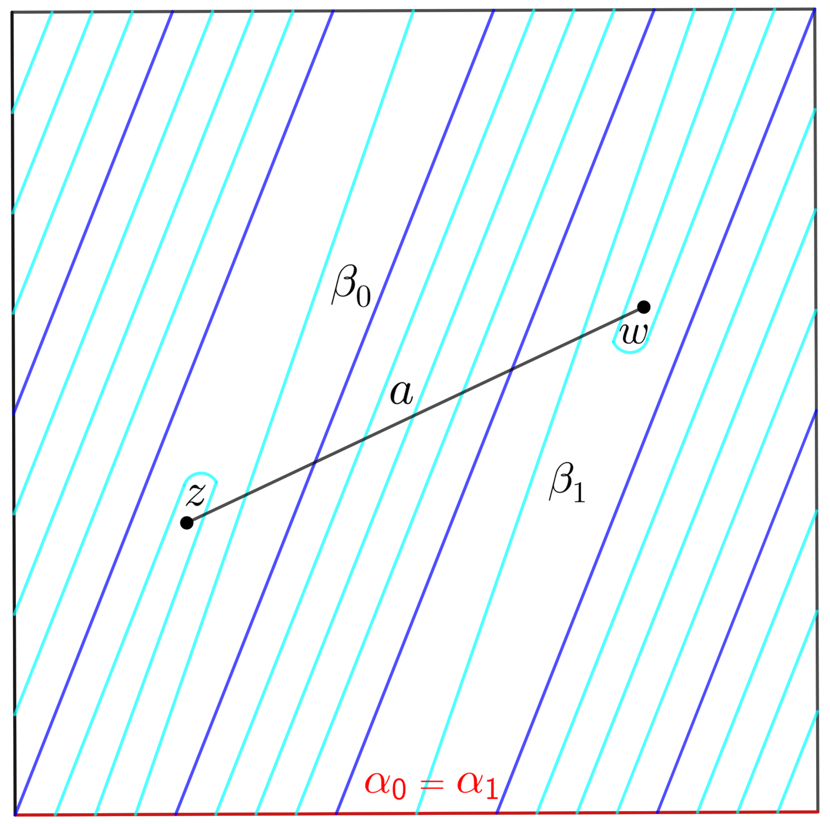
The left-handed trefoil knot is the 2-bridge knot $\mathfrak{b}(3,1)$.



Left: a doubly-pointed Heegaard diagram of the trefoil knot. By parameterization, it is $C(1,0,1,3,1)$.
Right: a doubly-pointed Heegaard diagram of a simple knot in the lens space $L(5,2)$. By parameterization, it is $C(5,3,2,1,0)$. Note that $L(5,2)\cong L(5,3)$.

Construction of a constrained knot

The right figure is a doubly-pointed Heegaard diagram $(T^2,\alpha_1,\beta_1,z,w)$ of a constrained knot K in the lens space $L(5,2)$. The Heegaard diagram (T^2,α_0,β_0) illustrates a standard Heegaard splitting of the lens space $L(5,2)$. The curve α_1 is the same as α_0 and the curve β_1 is chosen to be disjoint from β_0. Two basepoints z and w indicate how to contruct the knot $K=a\cup b$: choose an arc a (resp. b) in $T^2\backslash\alpha_1$ (resp. $T^2\backslash\beta_1$) connecting z to w and push it in the handlebody corresponding to α_1 (resp. β_1).

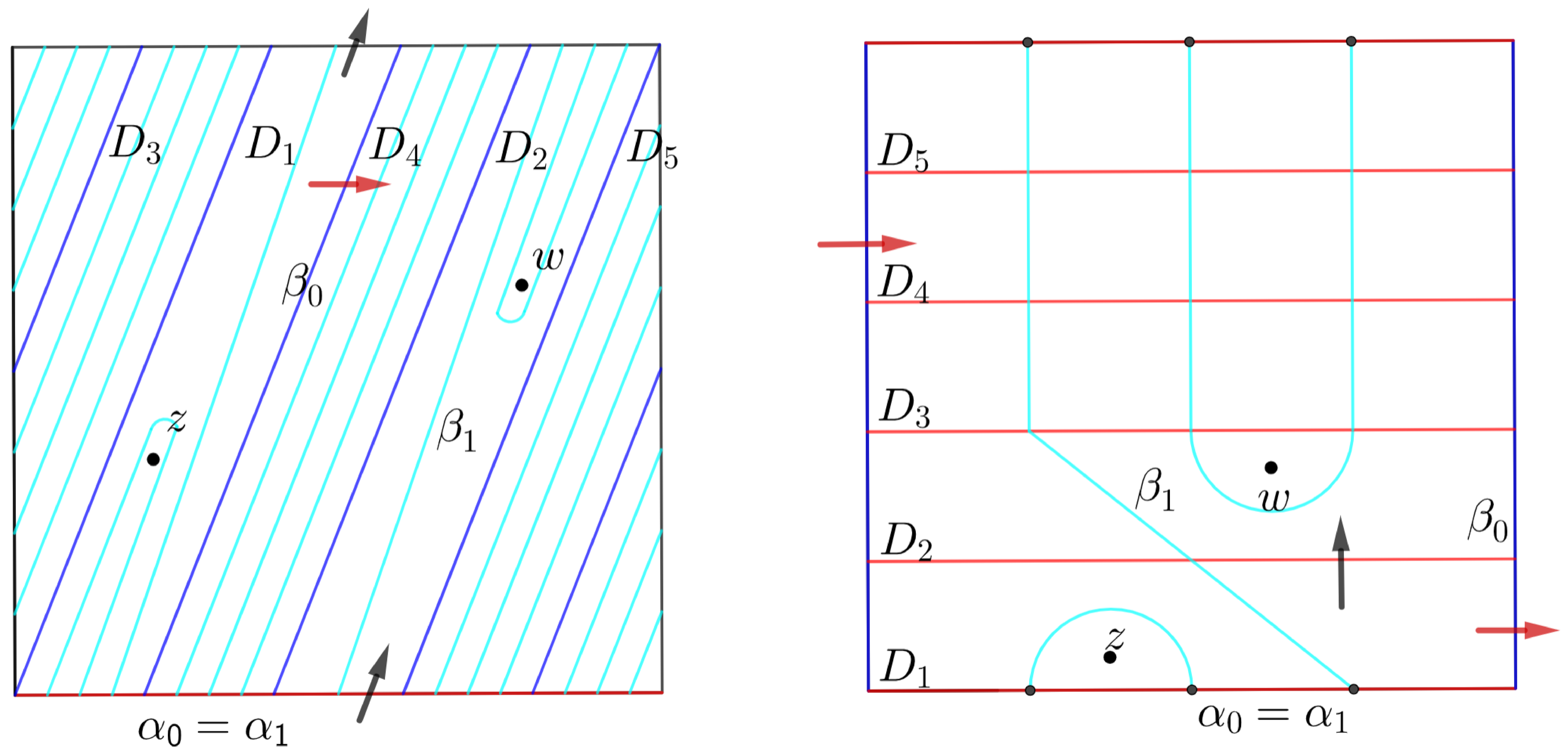


Main results

- [Ye20] There is a complete classification of constrained knots for the following parameterization.
- [LY20, BLY20, LY21] For a constrained knot K, its knot Floer homology $\widehat{HFK}(K)$ is isomorphic to its instanton knot homology $KHI(K)$, which are both determined by the Turaev torsion of K. The isomorphism $\widehat{HFK}(K)\cong KHI(K)$ can be generalized to torus knots in S^3 and lens spaces, and $(\pm 2,p,q)$ pretzel knots for odd integers p and q.

Parameterization: $C(p,q,l,u,v)$

Cut the diagram along β_0 and glue along α_0:



$p=5=$ the number of domains D_i;
$q=3$: the right-hand-side of $D_1$is glued to the left-hand-side of D_{1+q};
$l=2$: suppose $z\in D_1$; then $w\in D_l$;
$u=3=$ the number of intersection points of β_1 and the subarc of α_1;
$v=1=$ the number of rainbows around z.

Relation to orientable 1-cusped hyperbolic manifolds

Snappy program provides a list of 59068 orientable 1-cusped hyperbolic manifolds with at most 9 ideal tetrahedra. It can be verified that 21922 manifolds are complements of constrained knots, which, in particular, include the manifolds before $m130$. The full list can be found at http://faniel.wiki/about/. Note that the gap in the list (*e.g.* $m005$) is not orientable or not 1-cusped.

Name	Filling Slope & $C(p,q,l,u,v)$
$m003$	$(1,0)\&(10,3,3,1,0)$, $(-1,1)\&(5,4,5,3,1)$
	$(0,1)\&(5,4,5,3,1)$
$m004$	$(1,0)\&(1,0,1,5,2)$
$m006$	$(0,1)\&(15,4,2,1,0)$, $(1,0)\&(5,3,4,3,1)$
$m007$	$(1,0)\&(3,1,2,3,1)$
$m009$	$(1,0)\&(2,1,2,5,2)$
$m010$	$(1,0)\&(6,5,6,3,1)$
$m011$	$(1,0)\&(13,3,3,1,0)$, $(0,1)\&(9,4,9,3,1)$
$m015$	$(1,0)\&(1,0,1,7,2)$
...	...
$m130$	$(1,0)\&(16,3,6,1,0)$, $(0,1)\&(16,7,16,3,1)$
$m135$	Not from any constrained knot
...	...

References

[BLY20] John A. Baldwin, Zhenkun Li, and Fan Ye. Sutured instanton homology and Heegaard diagrams. *ArXiv: 2011.09424, v1*, 2020.

[LY20] Zhenkun Li and Fan Ye. Instanton Floer homology, sutures, and Heegaard diagrams. *ArXiv: 2010.07836, v2*, 2020.

[LY21] Zhenkun Li and Fan Ye. Instanton Floer homology, sutures, and Euler characteristics. *ArXiv: 2011.09424, v1*, 2021.

[Ye20] Fan Ye. Constrained knots in lens spaces. *ArXiv:2007.04237, v1*, 2020.