A large surgery formula for instanton Floer homology

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Joint work with Zhenkun Li
Basic idea

Knot Floer chain complex $\text{CFK}^\infty \cong \text{Heegaard Floer homology } \widehat{\text{HF}}(S^3_m(K))$.

Instanton knot homology $\text{KHI}$ but no differentials $\cong$ calculate $I^\#(S^3_m(K))$?

My work:

1. Construct $d_+$ and $d_-$ on $\text{KHI}$ analogous to $d_w$ and $d_z$ on $\text{CFK}^\infty$;
2. Use $d_+$ and $d_-$ to calculate $I^\#(S^3_m(K))$ for large integer $m$.

Conjecture (Kronheimer-Mrowka): $\text{KHI}(K) \cong \widehat{\text{HFK}}(K), I^\#(Y) \cong \widehat{\text{HF}}(Y)$.

Fact (Baldwin-Sivek): $\dim I^\#(Y) > |H_1(Y;\mathbb{Z})|$ implies the existence of irreducible $\text{SU}(2)$ representations of $\pi_1(Y)$.
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Suppose $Y$ is a closed 3-manifold and $\omega \to Y$ is a Hermitian line bundle with some admissible conditions. Based on Yang-Mills equations (related to $SO(3)$ connections), Floer ’88 constructed **instanton Floer homology** $I^\omega(Y)$.

Suppose $(M, \gamma)$ is a balanced sutured manifold, where $M$ is a 3-manifold with boundary and $\gamma \subset \partial M$ is a 1-submanifold with some balanced conditions. Kronheimer-Mrowka ’10 constructed **sutured instanton homology** $SHI(M, \gamma)$. 
Suppose $Y$ is a closed 3-manifold. Based on Heegaard diagrams and symplectic geometry, Ozsváth-Szabó ’04 constructed **Heegaard Floer homology** $\widehat{HF}(Y), HF^\infty(Y), HF^+(Y), HF^-(Y)$.

Suppose $K \subset Y$ is a knot. Ozsváth-Szabó ’04 and Rasmussen ’03 constructed **knot Floer homology** $HF^\circ(Y, K)$ for $\circ \in \{\hat{\prime}, \infty, +, -\}$.

Suppose $(M, \gamma)$ is a balanced sutured manifold. Juhász ’06 constructed **sutured Floer homology** $SFH(M, \gamma)$. 
### Special balanced sutured manifolds

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**Conjecture (Kronheimer-Mrowka ’10)**

\[ \text{SHI}(M, \gamma) \cong \text{SFH}(M, \gamma). \]

In particular, \( \text{KHI}(Y, K) \cong \widehat{\text{HFK}}(Y, K) \) and \( I^\#(Y) \cong \widehat{\text{HF}}(Y) \).

**Examples**

\( \text{KHI}(Y, K) \cong \widehat{\text{HFK}}(Y, K) \) holds for

- alternating links in \( S^3 \) (Kronheimer-Mrowka ’11)
- all torus knots (Li-Y. ’20 and Baldwin-Li-Y. ’20, some partial results by Lobb-Zentner ’13, Kronheimer-Mrowka ’14, Hedden-Herald-Kirk ’14, Daemi-Scaduto ’19, et al.)
- all \((1,1)\)-L-space knots and all constrained knots in lens spaces (Li-Y. ’21).
**Conjecture (Kronheimer-Mrowka ’10)**

\[ \text{SHI}(M, \gamma) \cong \text{SFH}(M, \gamma). \]

In particular, \( \text{KHI}(Y, K) \cong \widehat{\text{HF}}(Y, K) \) and \( I^\#(Y) \cong \widehat{\text{HF}}(Y). \)

**Examples**

\( I^\#(Y) \cong \widehat{\text{HF}}(Y) \) holds for

- \( \Sigma_2(S^3, L) \) for any alternating link \( L \) (Scaduto ’15);
- \( S^3_r(K) \) for any knot \( K \) admitting lens space surgeries. (Lidman-Pinzón-Scaduto ’20, Baldwin-Sivek ’20);
- Seifert fibered rational homology spheres (Alfieri-Baldwin-Dai-Sivek ’20);
- Strong Heegaard Floer L-spaces, i.e.
  \[ \dim \widehat{\text{HF}}(Y) = \dim \widehat{\text{CF}}(Y) = |H_1(Y; \mathbb{Z})| \] (Baldwin-Li-Y. ’20).
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4 Analogous constructions in instanton and Heegaard Floer theory
The hat version of the **bent complex** in Heegaard Floer theory:

For a knot $K \subset S^3$, choose a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$. Let $CFK^\infty(Y, K)$ be generated by $[x, i, j] \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \times \mathbb{Z} \times \mathbb{Z}$ with the Alexander grading $A(x) = j - i$ and let the differential be

$$
\partial[x, i, j] = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x, y) | \mu(\phi) = 1} \# \widehat{M}(\phi) \cdot [y, i - n_{w}(\phi), j - n_{z}(\phi)].
$$

Let $\widehat{A}_s$ be the subcomplex generated by $[x, i, j]$ with $\max\{i, j - s\} = 0$. 

![Diagram of bent complex]

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**Fan Ye (Cambridge)**

**Large surgery formula for KHI**

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Since \((\widehat{CF}(S^3), d_z) = \{i = 0\}\), \((\widehat{CF}(S^3), d_w) = \{j = 0\}\), let \(\hat{A}_s\) be generated by \(x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\) and let the differential \(d_s\) be

\[
d_s(x) = \begin{cases} 
  d_w(x) & A(x) > s, \\
  d_w(x) + d_z(x) & A(x) = s, \\
  d_z(x) & A(x) < s,
\end{cases}
\]

Fan Ye (Cambridge)
Theorem (large surgery formula, Oszváth-Szabó ’04, Rasmussen ’03)

For integer $m >> 0$ and any integer $s$ with $|s| \leq m/2$, there is an isomorphism

$$\widehat{HF}(S^3_m(K), [s]) \cong H(\hat{A}_s).$$

Here $[s] \in \mathbb{Z}/m$ is the corresponding spin$^c$ structure on $S^3_m(K)$.

Remark

The subcomplex $A^+_s$ generated by $[x, i, j]$ with $\max\{i, j - s\} \geq 0$ computes $HF^+(S^3_m(K), [s])$. 
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Main theorems

Theorem A (large surgery formula, Li-Y. '21)

There exist differentials $d_+$ and $d_-$ on $KHI(-S^3, K)$ so that

$$H(KHI(-S^3, K), d_+) \cong H(KHI(-S^3, K), d_-) \cong \mathcal{I}^\#(-S^3).$$

Define $A_s = (KHI(-S^3, K), d_s)$, where $d_s(x) = \begin{cases} 
  d_+(x) & A(x) > s, \\
  d_+(x) + d_-(x) & A(x) = s, \\
  d_-(x) & A(x) < s, 
\end{cases}$

For $m >> 0$ and any $s$ with $|s| \leq m/2$, there is an isomorphism

$$\mathcal{I}^\#(-S^3_{-m}(K), [-s]) \cong H(A_s).$$

Here $\mathcal{I}^\#(-S^3_{-m}(K)) = \bigoplus_{k=1}^{m} \mathcal{I}^\#(-S^3_{-m}(K), [k])$ is a spin$^c$-like decomposition.

*The minus sign comes from contact gluing maps (bypass maps).*
Main theorems

Theorem B (Li-Y. ’21)

If $K \subset S^3$ is an \textbf{instanton L-space knot}, then $\dim_{\mathbb{C}} KHI(S^3, K, i) \in \{0, 1\}$, where the $\mathbb{Z}/2$-gradings of the generators of $KHI(S^3, K, i) \cong \mathbb{C}$ are alternating. Hence there exists $k \in \mathbb{N}_+$ and integers $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$ so that

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{n_k} (-1)^{k-j} (t^{n_j} + t^{-n_j})$$

(from $\chi(KHI(K)) = \pm \Delta_K(t)$ by Lim ’09, Kronheimer-Mrowka ’10).

Remark

Oszváth-Szabó ’05 proved an analogous result for Heegaard Floer theory. The proof of Theorem B is inspired by their proof.
Main theorems

If $K$ is not an instanton L-space knot, then $\pi_1(S_r^3(K))$ has an irreducible $SU(2)$ representation for

1. all but finitely many slopes $r \in \mathbb{Q}\backslash\{0\}$ (Sivek-Zentner ’20);
2. $r = p/q$ with $p$ a prime power (Baldwin-Sivek ’19).

Corollary A (Li-Y. ’21)

The following knots are not instanton L-space knots.

1. Hyperbolic alternating knots (by Oszváth-Szabó ’05);
2. Montesinos knots (including all pretzel knots), except torus knots $T(2, 2n + 1)$, pretzel knots $P(-2, 3, 2n + 1)$ for $n \in \mathbb{N}_+$ and their mirrors (by Baker-Moore ’18).
3. Knots that are closures of 3-braids, except twisted torus knots $K(3, q; 2, p)$ with $pq > 0$ and their mirrors (by Lee-Vafaee ’21).
Main theorems

Theorem C (Baldwin-Li-Sivek-Y. 21)

For any nontrivial knot $K \subset S^3$, the group of the 3-surgery $\pi_1(S^3_3(K))$ has an irreducible $SU(2)$ representation.

Remark

Kronheimer-Mrowka '04 proved the existence of representation for slope in $[0, 2]$. Baldwin-Sivek '19 proved it for slope 4 and $p/q \in (2, 3)$ with $p$ a prime power. Theorem C is generalized to slope $p/q \in [16/5, 80/23] \cup (4, 5)$ with $p$ an odd prime power and $\gcd(p, 5) = 1$.

Theorem D (Li-Y. in preparation)

For any integer $n$, $I^\#(S^3_n(K))$ can be calculated by $d_+$ and $d_-$ on $KHI(-S^3, K)$ analogous to Oszváth-Szabó’s mapping cone formula for $\widehat{HF}(S^3_n(K))$. 
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Proposition A (surgery exact triangle, Floer '90, Scaduto '15)

Suppose $K$ is a knot in the interior of $M$. Let $(M_i, \gamma_i)$ be obtained from $(M, \gamma)$ by Dehn surgery along $K$ with slope $\mu_i$. If

$$\mu_1 \cdot \mu_2 = \mu_2 \cdot \mu_3 = \mu_3 \cdot \mu_1 = -1,$$

then there exists a long exact sequence

$$\text{SHI}(M_1, \gamma_1) \rightarrow \text{SHI}(M_2, \gamma_2) \rightarrow \text{SHI}(M_3, \gamma_3) \rightarrow \text{SHI}(M_1, \gamma_1)$$
Analogous constructions in instanton and Heegaard Floer theory

Let $K \subset S^3$ be a knot and let $M$ be the knot complement. Suppose $\mu$ and $\lambda$ are the meridian and the longitude of $K$. Let $\Gamma_n \subset \partial M$ be the suture consisting of two curves of slope $-n$ (i.e. $-n\mu + \lambda$). Push $\mu$ into $\text{int} M$ to obtain $\mu'$, with the framing induced by $\partial M$.

**Proposition A1 (Li-Y. 20)**

The $(\infty, 0, 1)$-surgery triangle on $\mu' \subset (-M, -\Gamma_n)$ induces

$$SHI(-M, -\Gamma_{n-1}) \rightarrow SHI(-M, -\Gamma_n) \rightarrow I^#(-S^3) \rightarrow SHI(-M, -\Gamma_{n-1})$$

(Note that $I^#(-S^3) \cong KHI(-S^3, \text{Unknot})$)

In general, let $(\hat{\mu}, \hat{\lambda}) = (\lambda - m\mu, -\mu)$ and let $\hat{\Gamma}_n$ be the suture consisting of two curves of $-n\hat{\mu} + \hat{\lambda}$. Then $(\infty, 0, 1)$-surgery triangle on $\hat{\mu'} \subset (-M, -\hat{\Gamma}_n)$ induces

$$SHI(-M, -\hat{\Gamma}_{n-1}) \rightarrow SHI(-M, -\hat{\Gamma}_n) \rightarrow I^#(-S^3_m(K)) \rightarrow SHI(-M, -\hat{\Gamma}_{n-1})$$
Analogous constructions in instanton and Heegaard Floer theory

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Proposition B (bypass exact triangle, Baldwin-Sivek ’18)

Suppose $\gamma_1, \gamma_2, \gamma_3$ are three sutures on $M$ such that $\gamma_i$ are the same except in a disk, where they look like as follows. Then there exists a long exact sequence

$$\text{SHI}(-M, -\gamma_1) \to \text{SHI}(-M, -\gamma_2) \to \text{SHI}(-M, -\gamma_3) \to \text{SHI}(-M, -\gamma_1)$$
Proposition B1 (Li-Y. 20)

Let $M = S^3 \setminus N(K)$ and let $\Gamma_\mu$ and $\Gamma_n$ be the sutures of slopes $\mu$ and $-n\mu + \lambda$. Then there are two bypass exact triangles

$$\xymatrix{ SHI(-M, -\Gamma_{n-1}) \ar[r]^{\psi_{+,n}^{n-1}} & SHI(-M, -\Gamma_n) \ar[r]^{\psi_{+,\mu}^n} & SHI(-M, -\Gamma_\mu) \ar[r]^{\psi_{+,n-1}^\mu} & }$$

Moreover, the bypass maps are homogeneous with respect to the Alexander gradings. Similarly, we can replace $\Gamma_{n-1}, \Gamma_n, \Gamma_\mu$ by $\hat{\Gamma}_{n-1}, \hat{\Gamma}_n, \hat{\Gamma}_\mu$. 
## Analogous constructions in instanton and Heegaard Floer theory

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Suppose $M$ is a 3-manifold with torus boundary. Based on Bordered Floer homology (Lipshitz-Oszváth-Thurston ’08), Hanselman-Rasmussen-Watson ’16 constructed a set of immersed curves in $\partial M \setminus \text{pt}$. It is denoted by $\widehat{HF}(M)$ and can be regarded as an object in some Fukaya category of $\partial M \setminus \text{pt}$. If $Y = M_1 \cup_{T^2} M_2$, then

$$\dim \widehat{HF}(Y) = \dim HF_{\text{sym}}(\widehat{HF}(M_1), \widehat{HF}(M_2)) = |\widehat{HF}(M_1) \cap \widehat{HF}(M_2)|.$$ 

In particular, when $M = S^3 \setminus N(K)$, we can recover $\widehat{HF}(S^3_r(K))$ and $\widehat{HFK}(S^3_r(K), K_r)$ as follows, where $K_r$ is the dual knot.
Immersed curves in the universal cover of $\partial M \cong T^2$

$$\dim \text{HFK}(T, i) = \begin{cases} \frac{1}{2} & z^2 = 1 \\ \frac{1}{2} & z^2 = -1 \\ \frac{1}{2} & z^2 = 0 \end{cases}$$

$$\dim \text{HFK}(S^3) = 1$$

$$\dim \text{HFK}(S^3(T)) = 1$$

$$\dim \text{HFK}(S^3(T), K_i, i) = \begin{cases} \frac{1}{2} & z^2 = 1 \\ \frac{1}{2} & z^2 = 0 \\ \frac{1}{2} & z^2 = -1 \end{cases}$$
Immersed curves in the universal cover of $\partial M \cong T^2$
Immersed curves in the universal cover of $\partial M \cong T^2$

\[
\hat{\mu} = -2\mu + \lambda \\
\hat{\lambda} = -\mu
\]

\[
-m + \hat{\lambda} = \mu - \lambda \\
-2\hat{\mu} + \hat{\lambda} = 3\mu - 2\lambda
\]
Immersed curves in the universal cover of $\partial M \cong T^2$

\[
\begin{align*}
\hat{\mu} &= -2\mu + \lambda \\
\hat{\lambda} &= -\mu \\
-\hat{\mu} + \hat{\lambda} &= \mu - \lambda \\
-2\hat{\mu} + \hat{\lambda} &= 3\mu - 2\lambda
\end{align*}
\]
Analogous constructions in instanton and Heegaard Floer theory

Note that all bypass maps are homogeneous with respect to Alexander gradings. Write $\Gamma_n$ for some grading summand of $SHI(-M, -\Gamma_n)$.

Define $d_{1,+} = \psi_{+,-\mu}^{n} \circ \psi_{+,n}^{\mu}$; $d_{2,+} = \psi_{+,-\mu}^{n-1} \circ (\psi_{+,n}^{n-1})^{-1} \circ \psi_{+,n}^{\mu}$; $d_{r,+} = \psi_{+,-\mu}^{n-r+1} \circ (\psi_{+,n-r+2}^{n-r+1})^{-1} \circ \cdots \circ (\psi_{+,n}^{n-1})^{-1} \circ \psi_{+,n}^{\mu}$. Then We have

1. $d_{r,+}$ is independent of $n$
2. $d_{r_1,+} \circ d_{r_2,+} = 0$ for any $r_1, r_2 \geq 1$, hence $d_+^2 = (\sum_r d_{r,+})^2 = 0$
3. $d_{r,+}$ increases the Alexander grading by $r$
4. $H(SHI(-M, -\Gamma_\mu), d_+) \cong I^\#(-S^3)$
Immersed curves in the universal cover of $\partial M \cong T^2$
Indeed, we have two spectral sequences associated to $d_{r,+}$ and $d_{r,-}$. Set $n = m$. Then we can construct $A_s$ as follows.
Step 1. Suppose $m >> 0$ and $\hat{\mu} = -m\mu + \lambda$. Then the slope of $\hat{\Gamma}_2$

$$-2\hat{\mu} + \hat{\lambda} = -2(-m\mu + \lambda) + (-\mu) = (2m - 1)\mu - 2\lambda$$

is large enough so that we can use 'middle Alexander gradings' of $SHI(-M, -\hat{\Gamma}_2)$ to recover the information of $I^\#(-S^3_{-m}(K), [s])$. 
Sketch of the proof of the large surgery formula

**Step 2.** The bypass exact triangle induces a long exact sequence

$$\to \text{SHI}(-M, -\Gamma_m) \xrightarrow{\psi^{\mu,-m-1} \circ \psi^{+,\mu}} \text{SHI}(-M, -\Gamma_{m-1}) \to \text{SHI}(-M, -\hat{\Gamma}_2) \to$$
Sketch of the proof of the large surgery formula

**Step 1.** Suppose $m >> 0$ and $\hat{\mu} = -m\mu + \lambda$. Then the slope of $\hat{\Gamma}_2$

$$-2\hat{\mu} + \hat{\lambda} = -2(-m\mu + \lambda) + (-\mu) = (2m - 1)\mu - 2\lambda$$

is large enough so that we can use 'middle Alexander gradings' of $SHI(-M, -\hat{\Gamma}_2)$ to recover the information of $I^\#(-S^3_{-m}(K), [s])$.

**Step 2.** The bypass exact triangle induces a long exact sequence

$$\cdots \rightarrow SHI(-M, -\Gamma_m) \xrightarrow{\psi^\mu_{-m-1} \circ \psi^m_{+\mu}} SHI(-M, -\Gamma_{m-1}) \rightarrow SHI(-M, -\hat{\Gamma}_2) \rightarrow \cdots$$

**Step 3.** Use the octahedral axiom (TR 4) to prove isomorphisms $H(A_s) \xrightarrow{\text{TR4}} H(\text{Cone}(\psi^\mu_{-m-1} \circ \psi^m_{+\mu})) \xrightarrow{\text{Step2}} SHI(-M, -\hat{\Gamma}_2, s') \xrightarrow{\text{Step1}} I^\#(-S^3_{-m}(K), [-s])$. 
Analogous constructions in instanton and Heegaard Floer theory

Further directions:

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Thanks for your attention.
Suppose $X, Y, Z, X', Y', Z'$ are graded spaces. Then three long exact sequences about $f, g, g \circ f$ induce the fourth one about $Z', Y', X'$.

\[ Z' = H(A_s) \]

\[ Y = \Gamma_{m-1} \oplus \Gamma_{m-1} \]

\[ Y' = \Gamma_m \]

\[ X = \Gamma_{\mu} \]

\[ X' = \Gamma_{m-1} \]

\[ f = (\psi_{+,m-1}, \psi_{-,m-1}) \]

\[ g = \mu_1 \]

\[ g \circ f = \psi_{+,m-1} \]

\[ \phi = \psi_{-,m-1} \circ \psi_{+,m} \]
Immersed curves in the universal cover of $\partial M \cong T^2$

Note: Fukaya category is also a triangulated category so also satisfies the octahedral axiom.
Analogous constructions in instanton and Heegaard Floer theory

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Analogous constructions in instanton and Heegaard Floer theory

**Theorem (Etnyre-Vela-Vick-Zarev ’17)**

The direct limit of the following system is isomorphic to $HFK^-(-S^3, K)$

$$SFH(-M, -\Gamma_{n-1}) \xrightarrow{\psi_{-,n}^{-1}} SFH(-M, -\Gamma_n) \xrightarrow{\psi_{-,n}^{n+1}} SFH(-M, -\Gamma_{n+1}) \xrightarrow{\psi_{-,n}^{n+2}}$$

The maps $\{\psi_{+,n-1}^n\}$ induce the $U$-action on $HFK^-(-S^3, K)$.

**Definition (Li ’19)**

Let $\text{KHI}^-(-S^3, K)$ be the direct limit of

$$SHI(-M, -\Gamma_{n-1}) \xrightarrow{\psi_{-,n}^{-1}} SHI(-M, -\Gamma_n) \xrightarrow{\psi_{-,n}^{n+1}} SHI(-M, -\Gamma_{n+1}) \xrightarrow{\psi_{-,n}^{n+2}}$$

Then the maps $\{\psi_{+,n-1}^n\}$ induce the $U$-action on $\text{KHI}^-(-S^3, K)$.

Moreover, we can replace $\Gamma_{n-1}, \Gamma_n, \Gamma_\mu$ by $\hat{\Gamma}_{n-1}, \hat{\Gamma}_n, \hat{\Gamma}_\mu$ to define $\text{KHI}^-(-S^3_{-m}(K), K_{-m})$ for the dual knot $K_{-m}$. 
Analogous constructions in instanton and Heegaard Floer theory

Note that for $s \ll 0$, we have $HF^{-}(S^{3}, K, s) \cong \hat{HF}(-S^{3})$ and $HF^{-}(S^{3}_{m}(K), K_{-m}, s) \cong \hat{HF}(-S^{3}_{m}(K), [s - s_0])$ for some $s_0$.

**Proposition (Li-Y. '20)**

For $s \ll 0$, we have

$$\bigoplus_{k=1}^{m} \text{KHI}^{-}(S^{3}_{-m}(K), K_{-m}, s + k) \cong I^{#}(-S^{3}_{-m}(K)).$$

Hence we can define $I^{#}(-S^{3}_{-m}(K), [s + k])$ by $\text{KHI}^{-}(S^{3}_{-m}(K), K_{-m}, s + k)$.

Since the direct system to define $\text{KHI}^{-}$ stabilizes for any fixed Alexander grading, we can also use 'middle gradings' of $SHI(-M, -\hat{\Gamma}_n)$ for any $n \gg 0$ to define the spin$^c$-like decomposition of $I^{#}(-S^{3}_{-m}(K))$. 
Diagram of the direct system

\[ -\Gamma_{n-1} \quad -\Gamma_n \quad -\Gamma_{n+1} \]

\[ \psi^{n-1}_{-,n} \quad \psi^n_{-,n+1} \]
### Analogous constructions in instanton and Heegaard Floer theory

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Theorem (Li-Y. 21)

For a balanced sutured manifold \((M, \gamma)\) with \(H = H_1(M; \mathbb{Z})\), we have a (possibly noncanonical) decomposition \(SHI(M, \gamma) = \bigoplus_{h \in H} SHI(M, \gamma, h)\). Define the Euler characteristic

\[
\chi(SHI(M, \gamma)) = \sum_{h \in H} \chi(SHI(M, \gamma, h)) \cdot h \in \mathbb{Z}[H]/\pm H.
\]

Then we have \(\chi(SHI(M, \gamma)) = \chi(SFH(M, \gamma)) = \tau(M, \gamma) \in \mathbb{Z}[H]/\pm H\).

Remark

The decomposition associated to the nontorsion part of \(H\) comes from the Alexander grading, and the torsion part comes from the 'middle gradings' of \(\Gamma_n\) for \(n \gg 0\).